Non-linear stability of Kerr–de Sitter black holes

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Einstein vacuum equations

$$\mathrm{Ein}(g) - \Lambda g = 0 \quad \Longleftrightarrow \quad \mathrm{Ric}(g) + \Lambda g = 0$$

- g: Lorentzian metric $(+ - - -)$ on 4-manifold Ω
- $\Lambda > 0$: cosmological constant
- $\mathrm{Ein}(g) = G_g \mathrm{Ric}(g)$, $G_g r = r - \frac{1}{2}(\mathrm{tr}_g\, r)g$ (trace-reversal)

Initial value problem

Data:

- Σ: 3-manifold
- h: Riemannian metric on Σ
- k: symmetric 2-tensor on Σ

Find spacetime (Ω, g), $\Sigma \hookrightarrow \Omega$, solving $\mathrm{Ric}(g) + \Lambda g = 0$, with

$$h = -g|_\Sigma, \quad k = II_\Sigma.$$

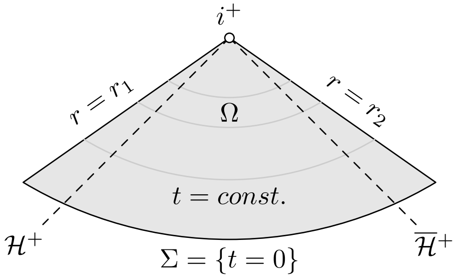
Theorem (Choquet-Bruhat '53)

Necessary and sufficient for local well-posedness: constraint equations for (h, k).

Key difficulty: diffeomorphism invariance $\Rightarrow$ need for gauge fixing

Kerr–de Sitter family

- manifold: $\Omega = [0,\infty)_t \times [r_1, r_2]_r \times \mathbb{S}^2$
- Cauchy surface: $\Sigma = \{t = 0\}$
- $\mathcal{C}^\infty$ family of stationary metrics g_b, $b = (M, \vec{a}) \in \mathbb{R} \times \mathbb{R}^3$
  - M: mass of the black hole
  - $\vec{a}$: angular momentum



Kerr–de Sitter family

Example. $b_0 = (M_0, \vec{0})$, Schwarzschild–de Sitter space

Metric:

$$g_{b_0} = f\,d\widetilde{t}^2 - f^{-1}\,dr^2 - r^2\,d\sigma^2, \quad f(r) = 1 - \frac{2M_0}{r} - \frac{\Lambda r^2}{3}$$

Valid for $r \in \big(r(\mathcal{H}^+, b_0), r(\overline{\mathcal{H}}^+, b_0)\big) \subset [r_1, r_2]$.

Extension: $t = \widetilde{t} - F(r)$

Black hole stability

Theorem (H.–Vasy '16)

Given $\mathcal{C}^\infty$ *initial data* (h, k) *on* Σ

- *satisfying the constraint equations,*
- *close (in* H^{21}*) to the initial data induced by* g_{b_0}*,*

there exist

- *a* $\mathcal{C}^\infty$ *metric* g *on* Ω *solving* $\mathrm{Ric}(g) + \Lambda g = 0$ *with initial data* (h, k) *at* Σ*,*
- *parameters* $b \in \mathbb{R}^4$ *close to* b_0 *such that*

$$g = g_b + \widetilde{g}, \quad |\widetilde{g}| = \mathcal{O}(e^{-\alpha t}),\ \alpha > 0.$$

Exponential decay towards a Kerr–de Sitter solution!

Related work

Non-linear stability:

- de Sitter: Friedrich ('80s), Anderson ('05), Ringström ('08), Rodnianski–Speck ('09)
- Minkowski: Christodoulou–Klainerman ('93), Lindblad–Rodnianski ('00s), Bieri–Zipser ('09), Speck ('14), Taylor ('15), Huneau ('15), LeFloch–Ma ('15)
- Hyperbolic space: Graham–Lee ('91)

Linear (mode) stability of black holes:

- $\Lambda > 0$: Kodama–Ishibashi ('04)
- $\Lambda = 0$: Regge–Wheeler ('57), Chandrasekhar ('83), Whiting ('89), with Andersson–Ma–Paganini ('16), Finster–Smoller ('00s+), Dafermos–Holzegel–Rodnianski ('16)

Related work

(Non-)linear fields on black hole spacetimes:

- $\Lambda > 0$: Bachelot ('91), Sá Barreto–Zworski ('97), Bony–Häfner ('08), Vasy ('13), Melrose–Sá Barreto–Vasy ('14), Dyatlov ('10s), Schlue ('15), H.–Vasy ('10s), ...
- $\Lambda = 0$: Wald ('79), Kay–Wald ('87), Andersson–Blue ('10s), Tataru, with Marzuola, Metcalfe, Sterbenz, and Tohaneanu ('10s), Luk ('13), Dafermos–Rodnianski–Shlapentokh-Rothman ('14), Lindblad–Tohaneanu ('16), ...

Gauge fixing

Goal: Solve $\mathrm{Ric}(g) + \Lambda g = 0$.

DeTurck[/Friedrich]: Demand that $\mathbf{1} \colon (\Omega, g) \to (\Omega, g_{b_0})$ be a [forced] wave map

$$\iff W(g) = g_{kl} g^{ij} (\Gamma(g)_{ij}^k - \Gamma(g_{b_0})_{ij}^k) dx^l = 0 \ [-\theta].$$

Reduced Einstein equation:

$$(\mathrm{Ric} + \Lambda)(g) - \delta_g^*(W(g)[+\theta]) = 0. \qquad (*)$$

- $(h, k) \mapsto$ Cauchy data for g in (*) with $W(g)[+\theta] = 0$ at Σ.
- Solve (*) (quasilinear wave equation) $\implies \mathcal{L}_{\partial_t} W(g) = 0$ at Σ.
- Second Bianchi: $\delta_g G_g \delta_g^*(W(g)[+\theta]) = 0$. Wave equation!

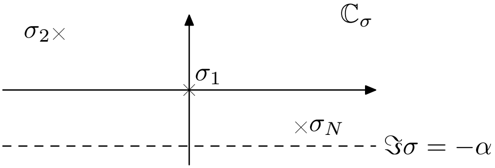
  $\implies W(g)[+\theta] = 0$, and $(\mathrm{Ric} + \Lambda)(g) = 0$.

Linearization around $g = g_{b_0}$

Asymptotics for equation $Lr = D_g(\mathrm{Ric} + \Lambda)(r) - \delta_g^* D_g W(r) = 0$:

$$r = \sum_{j=1}^{N} r_j a_j(x) e^{-i\sigma_j t} + \widetilde{r}(t,x), \quad t \gg 0,$$

- $\sigma_j \in \mathbb{C}$ resonances (quasinormal modes)
- $a_j(x)e^{-i\sigma t}$ resonant states, $L(a_j(x)e^{-i\sigma_j t}) = 0$
- $r_j \in \mathbb{C}$, $\widetilde{r} = \mathcal{O}(e^{-\alpha t})$, $\alpha > 0$ fixed, small

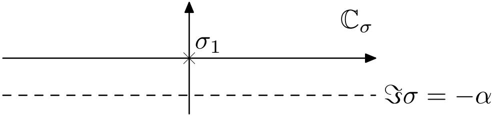


(Wunsch–Zworski '11, Vasy '13, Dyatlov '15, H. '15)

Attempt #1

$(Lr = 0,\ L = D_g(\mathrm{Ric} + \Lambda) - \delta_g^* D_g W.)$

Hope: $N = 1$, $\sigma_1 = 0$.



$$r = \frac{d}{ds} g_{b_0+sb}\big|_{s=0} + \widetilde{r}.$$

Then: Could solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta_g^* W(g_b + \widetilde{g}) = 0$$

for $\widetilde{g} = \mathcal{O}(e^{-\alpha t})$, $b \in \mathbb{R}^4$. (H.–Vasy '16)

False!

Attempt #2

$(Lr = 0,\ L = D_g(\mathrm{Ric} + \Lambda) - \delta_g^* D_g W.)$

Say $N = 2$, $\operatorname{Im} \sigma_2 > 0$, and (ignoring linearized KdS family)

$$r = r_2 a_2(x) e^{-i\sigma_2 t} + \widetilde{r}(t, x).$$

Hope: $a_2(x) e^{-i\sigma_2 t} = \mathcal{L}_V g$ is pure gauge, so $r = r_2 \mathcal{L}_{\chi V} g + \widetilde{r}$.

$$L\widetilde{r} = -L(r_2 \mathcal{L}_{\chi V} g) = r_2 \delta_g^* \underbrace{D_g W(\mathcal{L}_{\chi V} g)}_{=:\theta}$$

Thus:

$$D_g(\mathrm{Ric} + \Lambda)(\widetilde{r}) - \delta_g^* \big(D_g W(\widetilde{r}) + r_2 \theta \big) = 0.$$

Attempt #2, cont.

Then: Could solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta_g^*\big(W(g_b + \widetilde{g}) + \theta\big) = 0$$

for $\widetilde{g}$ and (b, θ) (finite-dimensional parameters).

Hope is false!

Not to worry: Have not yet used the constraint equations.

$$L = D_g(\mathrm{Ric} + \Lambda) - \delta_g^* D_g W$$

Delicate!

Fix?

$L = D_g(\mathrm{Ric} + \Lambda) - \delta^* D_g W$, where $\delta^* = \delta_g^* + 0$th order.

Apply linearized second Bianchi identity to

$$\begin{aligned} 0 &= L(a_2(x)e^{-i\sigma_2 t}) \\ &= D_g(\mathrm{Ric} + \Lambda)(a_2 e^{-i\sigma_2 t}) - \delta^* D_g W(a_2 e^{-i\sigma_2 t}). \end{aligned}$$

Fix?

$$\implies \delta_g G_g \delta^* \big(D_g W(a_2 e^{-i\sigma_2 t})\big) = 0.$$

Theorem (H.–Vasy '16)

For g the Schwarzschild–de Sitter metric, one can choose δ^* such that $\delta_g G_g \delta^*$ has no resonances with $\operatorname{Im}\sigma \geq 0$. ('Constraint damping,' Gundlach et al ('05).) Take

$$\delta^* w = \delta_g^* w + \gamma_1\, dt \otimes_s w - \gamma_2 w(\nabla t) g, \quad \gamma_1, \gamma_2 \gg 0.$$

$\implies D_g W(a_2 e^{-i\sigma_2 t}) = 0$

$\implies D_g(\mathrm{Ric} + \Lambda)(a_2 e^{-i\sigma_2 t}) = 0$

Mode stability for linearized Einstein equation (Kodama–Ishibashi '04) $\implies a_2 e^{-i\sigma_2 t} = \mathcal{L}_V g$

Attempt #3

Solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta^*\big(W(g_b + \widetilde{g}) + \theta\big) = 0$$

for $\widetilde{g} = \mathcal{O}(e^{-\alpha t})$ and (b, θ).

Almost: $\mathbf{1}$: $(\Omega, g_b) \to (\Omega, g_{b_0})$ is not a wave map: $W(g_b) \neq 0$.

Final attempt

Solve

$$(\mathrm{Ric} + \Lambda)(g_b + \widetilde{g}) - \delta^*\big(W(g_b + \widetilde{g}) - \chi W(g_b) + \theta\big) = 0.$$

Solution of the non-linear equation

Nash–Moser iteration scheme:

- Solve linearized equation globally at each step.
- Use linear solutions to update $\widetilde{g}$, b and θ.

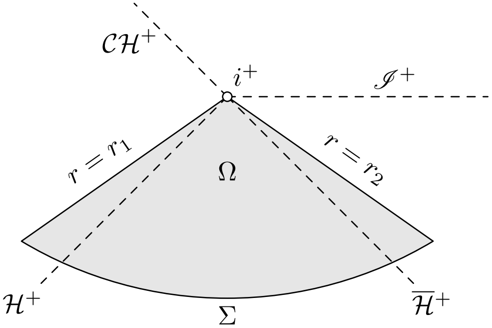
Automatically find

- final black hole parameters b,
- finite-dimensional modification θ of the gauge.

Outlook

- more precise asymptotic expansion, ringdown
- stability for large a (mode stability?)
- couple Einstein equation with matter
  - Non-linear stability of the Kerr–Newman–de Sitter family (for small a) for the Einstein–Maxwell system: work in progress
  - Einstein–(Maxwell–)scalar field, Einstein–Vlasov, ...

Outlook



- analysis in the cosmological region (ongoing work by Schlue)
- Penrose's Strong Cosmic Censorship Conjecture (work by Dafermos–Luk for Kerr)

Thank you!