Semi-Implicit Methods for All Mach Number Flow for the Euler Equations of Gas Dynamics

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Outline

All Mach number flow

- Compressible Euler equations of gas dynamics. The governing equations are rescaled so that the small Mach number ε (actually $\varepsilon = u_0/\sqrt{p_0/\rho_0}$) explicitly appears in the equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p = 0 \\ E_t + \nabla \cdot [(E + p)\mathbf{u}] = 0. \end{cases}$$

where ρ, is the fluid density, $\mathbf{u}$ is the velocity, E is the total energy density, and p is the pressure. The system is hyperbolic and the eigenvalues in direction $\mathbf{n}$ are: $\lambda_1 = \mathbf{u} \cdot \mathbf{n} - c/\varepsilon$, $\lambda_2 = \mathbf{u} \cdot \mathbf{n}$, $\lambda_3 = \mathbf{u} \cdot \mathbf{n} + c/\varepsilon$, $c^2 = (\partial p/\partial \rho)_s$

System is closed using the equation of state for ideal gases:

$$E = \frac{p}{\gamma - 1} + \frac{\varepsilon^2}{2} \rho |\mathbf{u}|^2$$

with $\gamma > 1$, being the ratio of specific heats.

All Mach number flow

In compressible flow, when Mach number is very small $\varepsilon \ll 1$, the flow speed is much less that the sound speed and the acoustic waves are much faster than material waves. In many cases such acoustic waves possess very small energy, and one is not interested in resolving them.

Difficulty: If classical explicit schemes are adopted, then for stability the time CFL time restriction is dictated by the sound speed:

$$\Delta t < \Delta x/\lambda_{\max} \quad \text{where } \lambda_{\max} = \max_{\Omega}(|u| + c_s/\varepsilon)$$

The numerical time step has to be much smaller than the typical macroscopic time scale of the problem, which becomes intolerable for $\varepsilon \ll 1$
In such cases on has to resort to implicit schemes.
But... *naive* implicit implementation of classical shock capturing schemes
⇒ excessive numerical viscosity for small Mach flows
⇒ solution is degraded.

Compressible vs incompressible flow

Large Mach number: $\varepsilon \approx 1$

- Compressible flow: hyperbolic systems of conservation laws
- shock discontinuities are generic
- conservation schemes are necessary (at least near discontinuity) in order to guarantee consistency for weak solutions
- CFL restriction is *physiological* since one is in general interested in acoustic waves

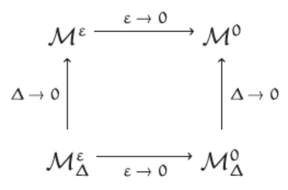
Small Mach number: $\varepsilon \ll 1$:

- quasi-incompressible flow
- often one is not interested in resolving acoustic waves
- material shocks do not form from smooth initial data and acoustic shocks have negligible amplitude
- classical CFL restriction is *pathological*: it is due to the stiffness of the problem and should be avoided

All Mach number flow

Framework

- Compressible and incompressible flows are usually treated by different techniques. Not obvious how to unify the treatment.
- Standard finite volume shock-capturing hyperbolic solvers have difficulties in the low Mach number regimes.
- Design numerical method for compressible flows that can handle both the compressible regime and the limit
- Remedy: use Asymptotic-Preserving (AP) methodology



First order time semi-discrete scheme

[In collaboration with: G. Russo., L. Scandurra.]
Consider the isentropic/ isothermal gas dynamics:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ (\rho \mathbf{u})_t + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho)/\varepsilon^2 = 0, \end{cases}$$

The system is closed by $p(\rho) = k\rho^\gamma$.

First order time semi-discrete scheme reads:

$$\begin{cases} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho \mathbf{u})^{n+1} = 0 \\ \frac{(\rho \mathbf{u})^{n+1} - (\rho \mathbf{u})^n}{\Delta t} + \nabla \cdot (\rho u \otimes u)^n + \nabla p(\rho)^{n+1}/\varepsilon^2 = 0, \end{cases}$$

As $\varepsilon \to 0$ the scheme becomes (formally) a consistent discretization of incompressible Euler equations:

$$\rho^{n+1} = \rho_0. \quad \nabla \cdot \mathbf{u}^{n+1} = 0, \quad \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u})^n + \nabla \pi^{n+1} = 0,$$

GOAL

- Construction of staggered central schemes for compressible Euler **that are able to capture** the incompressible Euler limit as $\varepsilon \to 0$.
- Shock Capturing scheme ($\Rightarrow$ Consevative) for large ε.
- CFL condition independent of ε
- Low dissipation for low ε.

Why staggered?

GOAL

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Why staggered? Simplicity.
With staggered central schemes (Nessyahu-Tadmor type):
• not care for numerical flux
• the scheme is automatically central and compact for the implicit terms.

Features of the scheme:

- Spatial discretization: Second-order, nonoscillatory central scheme on a staggered grid, NT central scheme.
- Time Integrator: Semi-implicit approach based on IMEX Runge-Kutta methods.

IMEX scheme for isentropic gas dynamics on staggered grid

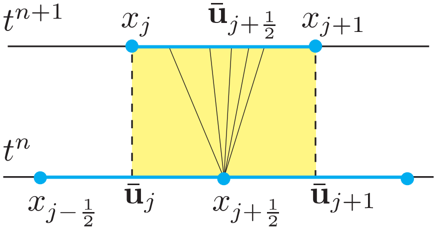
Consider the equations for isentropic gas dynamics in 1D:

$$\rho_t + m_x = 0$$

$$m_t + \left(\frac{m^2}{\rho} + \frac{p}{\varepsilon^2}\right)_x = 0$$

and $p = \rho^\gamma$.

Integrate the equation on a staggerd grid, from time t^n to t^{n+1}



First order in time and NT scheme for isentropic Euler

$$\begin{cases} \bar{\rho}^{n+1}_{j+1/2} = \bar{\rho}^{n}_{j+1/2} - \dfrac{\Delta t}{\Delta x}(m^{n+1}_{j+1} - m^{n+1}_{j}) \\ \bar{m}^{n+1}_{j+1/2} = \bar{m}^{n}_{j+1/2} - \dfrac{\Delta t}{\Delta x}(f^{n}_{j+1} - f^{n}_{j}) - \dfrac{\Delta t}{\epsilon^2 \Delta x}(p^{n+1}_{j+1} - p^{n+1}_{j}) \end{cases}$$

where $f^n_j = (\bar{m}^n_j)^2/\bar{\rho}^n_j$ and $\bar{\rho}_{j+1/2} = \frac{\bar{\rho}_{j+1}+\bar{\rho}_j}{2} + \frac{1}{8}(\rho'_j - \rho'_{j+1})\Delta x$

$$m^{n+1}_j = m^n_j - \frac{\Delta t}{\Delta x}(f^n)'_j - \frac{\Delta t}{\varepsilon^2 \Delta x}(p^{n+1})'_j$$

where: ρ'_j, $(f^n)'_j$ and, $(p^{n+1})'_j$ are a first order approximation of the space derivatives on cell j (e.g. MinMod).
Using equation for m^{n+1}_j, and plugging it in the equation for $\bar{\rho}^{n+1}_{j+1/2}$ one obtains an equation of the form:

$$\bar{\rho}^{n+1}_{j+1/2} = \rho^{*}_{j+1/2} + \frac{\Delta t^2}{\epsilon^2 \Delta x^2}\left(p^{n+1}_{j+3/2} - 2p^{n+1}_{j+1/2} + p^{n+1}_{j-1/2}\right)$$

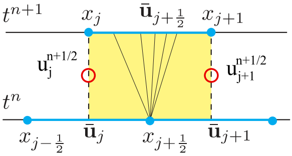
$\rho^{*}_{j+1/2}$ are computed explicitly (in a conservative way)
Considering that $p = p(\rho)$, this is a non linear system.
It can be effectively solved by either

- considering $\rho = \rho(p) \Rightarrow$ nonlinearity is only in the diagonal $\rightarrow$ system can be effectively solved by few Newton iterations.
- approximating the discrete Laplacian in p at time t^n by $D_x(p'(\rho^n)D_x\rho^{n+1})$ thus obtaining a linearly implicit scheme

Second order extension in time

The scheme is already second order in space.
Second order in time: proceed in a way similar to NT scheme:

- compute a semi-implicit predictor (non necessarily conservative) at location $\{x_j, t^{n+1/2}\}$ for $\{u_j^{n+1/2}\}$
- compute a conservative correction for $\{\bar{u}_{j+1/2}^{n+1}\}$



The technique can in principle be extended to s-stage RK schemes:

- compute a semi-implicit predictor at location $\{x_j, t^{n+c_i\Delta t}\}$ for the (non conservative) stage values $\{U_j^{(i)}\}$, $i = 1, \ldots, s-1$
- compute a conservative correction for $\{\bar{u}_{j+1/2}^{n+1}\}$

isentropic gas dynamics

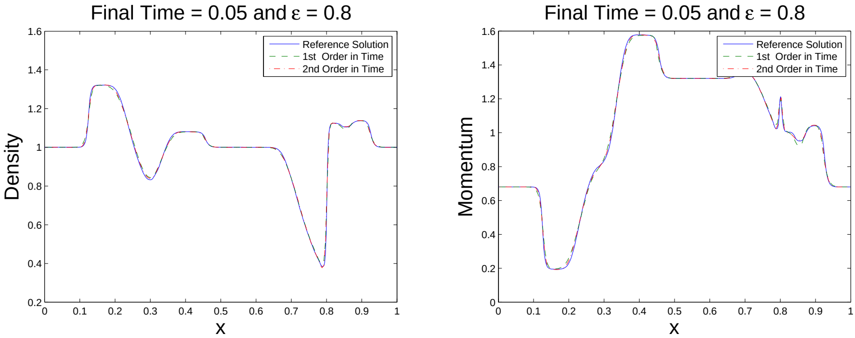
Consider the 2×2 system of Euler equation, with the initial conditions[1]

$$\begin{cases} \rho(x,0) = 1.0, m(x,0) = 1 - \frac{\varepsilon^2}{2}, & x \in [0,0.2] \bigcup [0.8,1], \\ \rho(x,0) = 1 + \varepsilon^2, m(x,0) = 1, & x \in [0,0.2], \\ \rho(x,0) = 1, m(x,0) = 1 + \frac{\varepsilon^2}{2}, & x \in [0.3,0.7], \\ \rho(x,0) = 1 - \varepsilon^2, m(x,0) = 1, & x \in [0.7,0.8], \end{cases} \quad (1)$$

System closed by $p(\rho) = \rho^2$.
Here, we adopt different value of $\varepsilon = 0.8,\ 0.3,\ 0.05$.
We fix the time step as used in the literature.

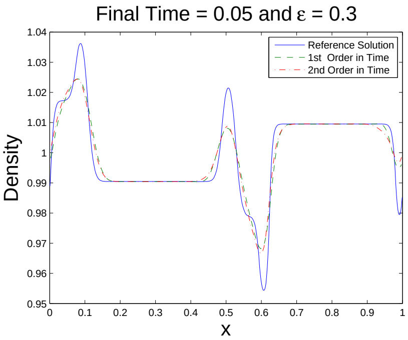
[1] Degond and Tang, 2011

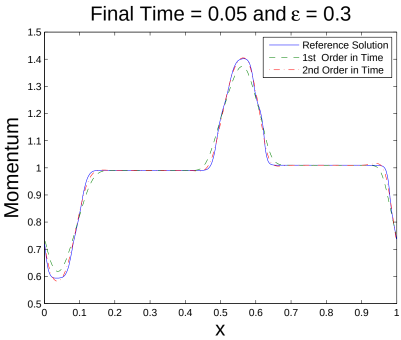
Numerical results for first and second order IMEX RK schemes



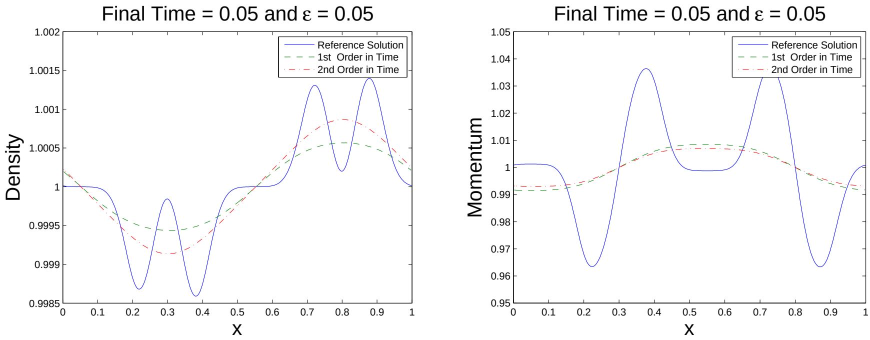
Numerical results at time $T = 0.05$ with $\Delta x = 1/200, \Delta t = 1/2000$, for the density (Left) and momentum (Right) for $\epsilon = 0.8$. The solid line is the reference solution computed with $\Delta x = 1/500$ and $\Delta t = 1/20000$.

Numerical results for scheme IMEX





Numerical results for scheme IMEX-I



The results are very close to those obtained by Degond and Tang (2011).

Complete Euler equations

Consider the compressible Euler equations.
Assume a polytropic gas with constant $\gamma \geq 1$.

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho \mathbf{u}) & = 0 \\
(\rho \mathbf{u})_t + \nabla \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho} + \frac{p \cdot I}{\varepsilon^2} \right) & = 0 \\
E_t + \nabla \cdot ((E + p)\mathbf{u}) & = 0. \\
p = (\gamma - 1)E - (\gamma - 1)\epsilon^2 \rho u^2 / 2 &
\end{cases}
$$

First order IMEX + pressure equation gives:[2].

First order Semi-Implicit discretization

$$
\begin{cases}
\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot \mathbf{m}^{n+1} & = 0, \\
\frac{m^{n+1} - m^n}{\Delta t} \Delta t + \nabla \cdot \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right)^n - \frac{\gamma - 1}{2} \nabla \left(\frac{|\mathbf{m}|^2}{\rho} \right)^n + \frac{\gamma - 1}{\varepsilon^2} \nabla E^{n+1} & = 0, \\
\frac{E^{n+1} - E^n}{\Delta t} \Delta t - \nabla \cdot \left(\frac{\gamma - 1}{2} \varepsilon^2 \frac{|\mathbf{m}|^2 \mathbf{m}}{\rho^2} \right)^n + \nabla \cdot (\gamma \frac{E^n}{\rho^n} \mathbf{m}^{n+1}) & = 0, \\
p^{n+1} = (\gamma - 1) \left(E^{n+1} - (\gamma - 1)\epsilon^2 (\rho u^2)^n / 2 \right). &
\end{cases}
$$

[2] Di Marco, Loubere, Vignal, 2016

Complete Euler equations in 1D

Similarly ad isentropic case, we discretize Euler equations in 1D on a staggered grid, with a first order SI-IMEX method in time, obtaining:

$$
\begin{aligned}
\bar{\rho}^{n+1}_{j+1/2} &= \bar{\rho}^{n}_{j+1/2} & & -\frac{\Delta t}{\Delta x}\left(m^{n+1}_{j+1} - m^{n+1}_{j}\right) \\
\bar{m}^{n+1}_{j+1/2} &= \bar{m}^{n}_{j+1/2} & -\frac{\Delta t}{\Delta x}\frac{3-\gamma}{2}\left(f^{n}_{j+1} - f^{n}_{j}\right) & -\frac{\Delta t}{\Delta x}\frac{\gamma-1}{\epsilon^2}\left(E^{n+1}_{j+1} - E^{n+1}_{j}\right) \\
\bar{E}^{n+1}_{j+1/2} &= \bar{E}^{n}_{j+1/2} & -\frac{\Delta t}{\Delta x}\left(g^{n}_{j+1} - g^{n}_{j}\right) & -\gamma\frac{\Delta t}{\Delta x}\left(\frac{E^{n}_{j+1} m^{n+1}_{j+1}}{\rho^{n}_{j+1}} - \frac{E^{n}_{j} m^{n+1}_{j}}{\rho^{n}_{j}}\right)
\end{aligned}
$$

Here for short we denoted $f = \rho u^2$ and $g = -(\gamma-1)\rho u^3/2$.
Just as in the case of isentropic gas dynamics, an equation for m^{n+1}_j is adopted and its expression is substituted in the equation for the energy obtaining:

$$\bar{E}^{n+1}_{j+1/2} = E^{*}_{j+1/2} + \gamma(\gamma-1)\frac{\Delta t^2}{\epsilon^2 \Delta x^2} \cdot \cdot \left(\frac{E^n_{j+1}}{\rho^n_{j+1}} E^{n+1}_{j+3/2} - \left(\frac{E^n_{j+1}}{\rho^n_{j+1}} + \frac{E^n_j}{\rho^n_j} \right) E^{n+1}_{j+1/2} + \frac{E^n_j}{\rho^n_j} E^{n+1}_{j-1/2} \right)$$

where, as usual, by $E^{*}_{j+1/2}$ denotes something that can be computed explicitly. This results in the solution of a simple tridiagonal system.

- The use of SI-IMEX scheme provides effective solution with no iterative solver.
- Second order extension in time is obtained by using a suitable GSA IMEX scheme

Remark Although intermediate stages can be computed at position x_j, even with non conservative schemes, evidence shows that the last stage has to be computed on the numerical solution $\bar{u}^{n+1}_{j+1/2}$. This enforces the use of a Globally Stiffly Accurate RK-IMEX scheme.

Two colliding acoustic pulses problem[3]

Two colliding acoustic pulses in a weakly compressible regime.
The domain is $-L \leqslant x \leqslant L = 2/\varepsilon$ and the initial data are given by

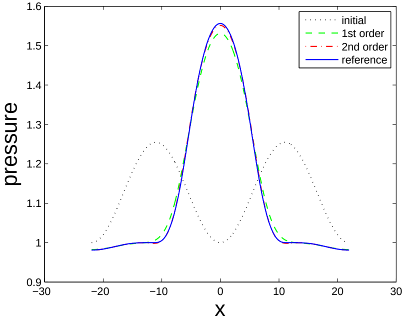
$$
\begin{aligned}
\rho(x,0) &= \rho_0 + \tfrac{1}{2}\varepsilon\rho_1\left(1-\cos\left(\tfrac{2\pi x}{L}\right)\right), \quad \rho_0 = 0.955, \quad \rho_1 = 2.0,\\
u(x,0) &= \tfrac{1}{2}u_0\,\mathrm{sign}(x)\left(1-\cos\left(\tfrac{2\pi x}{L}\right)\right), \quad u_0 = 2\sqrt{\gamma},\\
p(x,0) &= p_0 + \tfrac{1}{2}\varepsilon p_1\left(1-\cos\left(\tfrac{2\pi x}{L}\right)\right), \quad p_0 = 1.0, \quad p_1 = 2\gamma,
\end{aligned} \tag{2}
$$

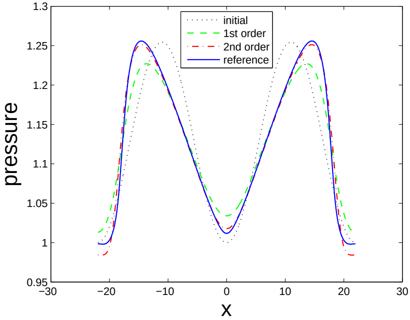
We tested $\varepsilon = 1/11$ and and $\varepsilon = 10^{-4}$. Number of cells $N = 440$.

[3] Noelle, Bispen, Arun, Lukácová-Medvidová and Munz, 2014

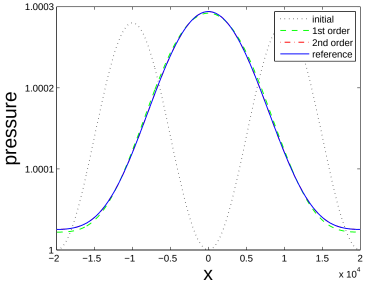
Complete Euler: results

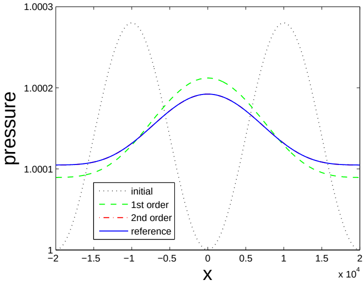
Second order in time and space
Time step set by (almost) material CFL:
$\Delta t = \mathrm{CFL} * \Delta x / u_{\max}$, $u_{\max} = \max |u| + c_s$. Periodic BC.
Left: time $T = 0.815$, Right time $T = 1.63$, $\varepsilon = 1/11$.





Left: time $T = 0.815$, Right: time $T = 1.63$ and $\varepsilon = 10^{-4}$.





2D convergence test

Moving vortex with compact support vorticity

Test that does not contain acoustic waves.

$\Rightarrow$ We may be accurate also with material time steps no need to resolve ε in order to observe convergence.

$$\begin{cases} \rho(x,y,0) = 110 + \left(\frac{1.5\varepsilon}{4\pi}\right)^2 (k(4\pi r) - k(\pi))\chi_{4r\leqslant 1}, \\ u(x,y,0) = 0.6 + 1.5(1+\cos(4\pi r))(0.5-y)\chi_{4r\leqslant 1}, \\ v(x,y,0) = 1.5(1+\cos(4\pi r))(x-0.5)\chi_{4r\leqslant 1} \end{cases}$$

where $r = \|\mathbf{x} - (0.5, 0.5)^T\|$, Mach $M = 0.6\varepsilon/\sqrt{110}$,

$$k(r) = 2\cos(r) + 2r\sin(r) + \frac{\cos(2r)}{8} + \frac{r}{4}\sin(2r) + 0.75r^2.$$

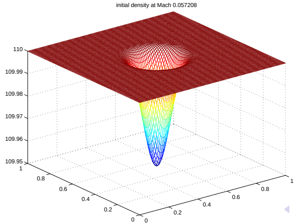


Table of convergence

$M = 0.1, \; T = 0.1$						
N	L^1-error ρ	EOC ρ	L^1-error m	EOC m	L^1-error E	EOC E
10	1.55e-04		2.44e-03		2.17e-04	
20	8.66e-05	0.842	1.06e-03	1.199	1.21e-04	0.8431
40	1.95e-05	2.145	2.47e-04	2.106	2.73e-05	2.1450
80	4.22e-06	2.213	6.09e-05	2.021	5.90e-06	2.2130
160	9.26e-07	2.188	1.65e-05	2.188	1.29e-06	2.1882

$M = 0.01, \; T = 0.1$						
N	L^1-error ρ	EOC ρ	L^1-error m	EOC m	L^1-error E	EOC E
10	3.80e-05		2.50e-03		5.33e-05	
20	2.33e-05	0.708	1.10e-03	1.178	3.26e-05	0.708
40	7.98e-06	1.546	2.53e-04	2.128	1.11e-05	1.546
80	1.99e-06	1.999	6.17e-05	2.035	2.79e-06	1.999
160	4.82e-07	2.048	1.66e-05	2.048	6.75e-07	2.048

Numerical Asymptotic preserving (AP) property

AP property

A numerical scheme applied to the compressible Euler equations is said to be AP if in the limit as $\varepsilon \to 0$, such scheme provides a consistent discretization of the incompressible Euler equation.

2D incompressible Euler equation, vorticity stream-function formulation:

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0, \quad (x, y) \in [0, 2\pi] \times [0, 2\pi]$$

Periodic boundary conditions and as initial conditions,

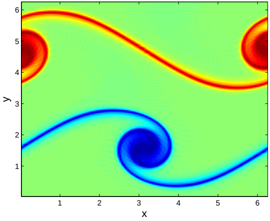
$$\omega(x, y, 0) = \begin{cases} \delta \cos(x) - \frac{1}{\rho} sech^2((y - \pi/2)/\rho), & y \leqslant \pi, \quad \delta = 0.05, \quad \rho = \frac{\pi}{15} \\ \delta \cos(x) + \frac{1}{\rho} sech^2((3\pi/2 - y)/\rho), & y > \pi \end{cases}$$

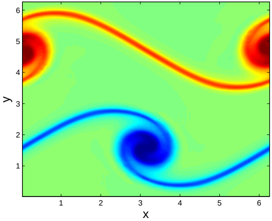
Reference solution is obtained by spectral method + RK4 at Time $T = 6$.

Numerical Asymptotic preserving (AP) property

Second order semi-Implicit central scheme for compressible Euler equations.
We set $\varepsilon = 10^{-4}$, periodic BC, $N = 160$ and $T = 6.0$.

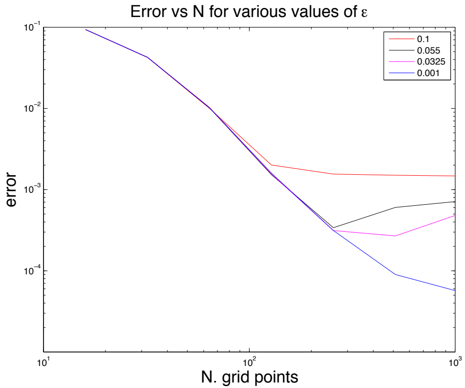
Numerical solution for 2D Euler:
Reference solution (left), numerical solution (compressible cas) $\varepsilon = 10^{-4}$ (right).





Numerical Asymptotic preserving (AP) property

Behaviour of the error in L^1-norm for small values of Mach number:
$\varepsilon = 0.1, 0.055, 0.0325 \cdot 10^{-2}, 0.001$
and $N = 16, 32, 64, \cdots, 1024$



A formal proof of the AP property of the scheme is reported in a paper in preparation [S.Boscarino, G.R., L.Scandurra]

Non staggered schemes.

[in collaboration with A. Iollo and F. Bernard]
Non staggered schemes are usually preferred in several applications.

Goal: construct schemes that are simple enough in view of multidimensional flow, and compare them on a wide range of Mach numbers.

Two family of schemes are considered:

Pressure schemes:
- explicit treatment of convection terms
- implicit treatment of pressure terms

Flux splitting:
Flux is split using characteristics, into “material” and “acoustic” flux
- explicit treatment of material terms
- implicit treatment of acoustic terms

In all schemes, space derivatives of implicit terms, D_x, are central.
Space derivative of explicit terms, $\hat{D}_x$, are treated by various numerical fluxes. In particular

upwind on the fluid velocity $\mathbf{u}$ and local Lax-Friedrichs.

Pressure splitting

$$\begin{cases} \rho^{n+1} = \rho^n - \Delta t \hat{D}_x(m^{n+1}) \\ m^{n+1} = m^n - \Delta t \hat{D}_x(m^n u^n) - \Delta t D_x(p^{n+1}) \\ E^{n+1} = E^n - \Delta t D_x(h^n m^{n+1}) \end{cases} \quad (3)$$

where $h = e + p/\rho$ is the total enthalpy density.
Inserting the expression for m^{n+1} into the equation for energy one obtains

$$E^{n+1} = E^n - \Delta t \hat{D}_x(h^n m^*) + \Delta t^2 D_x(h^n D_x(p^{n+1}))$$

where $m^* = m^n - \Delta t \hat{D}_x(m^n u^n)$.
Using the equation of state $\frac{p^{n+1}}{\gamma - 1} = E^{n+1} - \frac{m^{n+1} u^n}{2}$ we get

$$p^{n+1} = (\gamma - 1)\Big(E^* + \Delta t^2 D_x(h^n D_x(p^{n+1})) + \Delta t \frac{u^n}{2} D_x(p^{n+1})\Big) \quad (4)$$

where $E^* = E^n - \Delta t \hat{D}_x(h^n m^*) - \frac{m^* u^n}{2}$.

Pressure splitting: resolution on Energy

Similar system can be written in terms of energy.

Using $p^{n+1} = (\gamma - 1)(E^{n+1} - \frac{m^n u^n}{2})$ in momentum equation we obtain

$$m^{n+1} = m^n - \frac{3-\gamma}{2}\Delta t \hat{D}_x(m^n u^n) - (\gamma - 1)\Delta t D_x(E^{n+1}) \qquad (5)$$

Plugging this expression in the equation on the energy

$$E^{n+1} = E^* + (\gamma - 1)\Delta t^2 D_x(h^n D_x(E^{n+1})) \qquad (6)$$

with $E^* = E^n - \Delta t \hat{D}_x(h^n m^*)$ and $m^* = m^n - \frac{3-\gamma}{2}\Delta t \hat{D}_x(m^n u^n)$.

E^{n+1} can now be computed and plugged into the equation of momentum to find m^{n+1}.

Flux splitting

Make use of the special property of Euler equations:

$$F(U) = A(U)U$$

where $A(U) = \nabla_U F(U)$.
Let $A = Q\Lambda Q^{-1}$, where $\Lambda = \mathrm{diag}(u-c, u, u+c)$, Q: right eigenvectors of A.
Then write $A = A_u + A_c$, where $A_u = Q\Lambda_u Q^{-1}$, and $A_c = Q\Lambda_c Q^{-1}$
Then $F = F_u + F_c$, with $F^u = A_u U$, and $F_c = A_c U$.
F_u is treated upwind, while F_c is treated central semi-implicit.
The first order scheme becomes:

$$\begin{cases} \rho^{n+1} = \rho^n - \dfrac{\gamma-1}{\gamma}\Delta t \hat{D}_x(m^n) - \dfrac{\Delta t}{\gamma} D_x(m^{n+1}) \\ m^{n+1} = m^n - \dfrac{\gamma-1}{\gamma}\Delta t \hat{D}_x(m^n u^n) - \dfrac{\Delta t}{\gamma} D_x(m^n u^n) - \Delta t\, D_x(P^{n+1}) \\ E^{n+1} = E^n - \dfrac{\gamma-1}{\gamma}\Delta t \hat{D}_x(\rho^n (u^n)^3) - \dfrac{\Delta t}{\gamma} D_x(\rho^n (u^n)^3) - \Delta t \dfrac{\gamma}{\gamma-1} D_x(\dfrac{p^n}{\rho^n} m^{n+1}) \end{cases}$$

Flux splitting: resolution on pressure

Plug the expression of m^{n+1} in the equation on the energy

$$E^{n+1} = E^* + \Delta t^2 \frac{\gamma}{\gamma - 1} D_x\left(\frac{p^n}{\rho^n} D_x(p^{n+1})\right)$$

with E^* explicitly computed.
Let us rewrite the energy at time t^{n+1} as

$$E^{n+1} = \frac{p^{n+1}}{\gamma - 1} + \frac{m^n u^n}{2}$$

Thus

$$\frac{p^{n+1}}{\gamma - 1} = E^{**} + \Delta t^2 \frac{\gamma}{\gamma - 1} D_x\left(\frac{p^n}{\rho^n} D_x(p^{n+1})\right) \tag{7}$$

where E^{**} can be computed explicitly.
Once p^{n+1} is known then all other quantities can be computed.

A similar procedure can be obtained by selecting E^{n+1} as basic unknown.

Second order schemes

Second order in space

Central difference is automatically second order. Upwind difference are made second order by a classical limiter on derivatives:

$$u_j' = \mathrm{MimMod}\left(\theta\frac{\bar{u}_{j+1}-\bar{u}_j}{\Delta x}, \frac{\bar{u}_{j+1}-\bar{u}_{j-1}}{2\Delta x}, \theta\frac{\bar{u}_j-\bar{u}_{j-1}}{\Delta x}\right),$$

Second order in time

It is obtained by using IMEX schemes on

$$U' = \mathcal{H}(U, U)$$

The first argument is treated explicitly, the second one implicitly.
Here we used the scheme

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \hat{c} & \hat{c} & 0 \\ \hline & 1-\alpha & \alpha \end{array} \qquad \begin{array}{c|cc} \alpha & \alpha & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array} \tag{8}$$

where $\alpha = 1 - 1/\sqrt{2}$ and $\hat{c} = 1/(2\alpha)$.
It is a combination of explicit RK2 and L-stable SDIRK.

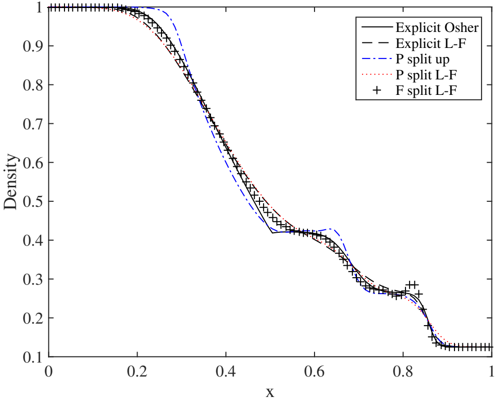
Numerical tests

Two sets of tests have been adopted:

Classical shock tube problems
to test the capability of capture shocks

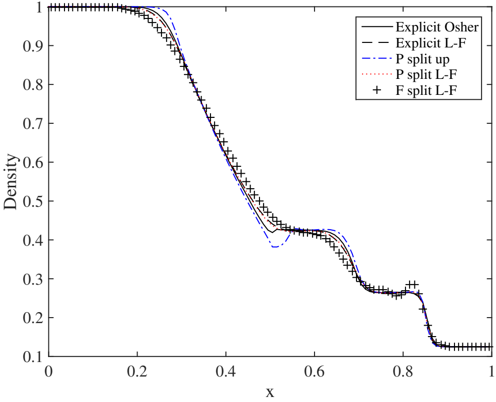
Two acoustic pulses
To check the capability to filter out acoustic waves at small Mach numbers

Sod shock tube problem - first order



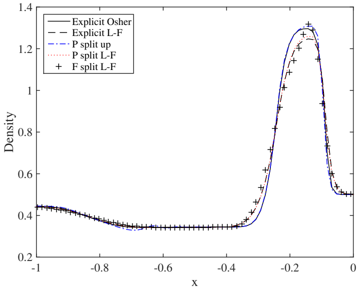
Small oscillations appear in the F-split scheme

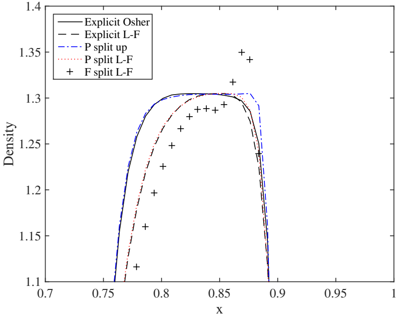
Sod shock tube problem - second order



Oscillations appear also in the P split-upwind,
while p-split+LLF is comparable with explicit+LLF

Lax shock tube problem - second order



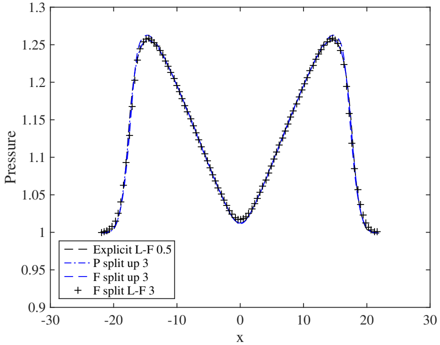


- F-split LLF shows oscillations.
- Explicit Osher and P-split up are comparable.
- Explicit LLF and P-split LLF are comparable.

Acoustic waves

Test taken from Cordier, Degond, Kumbaro, JCP 2012.

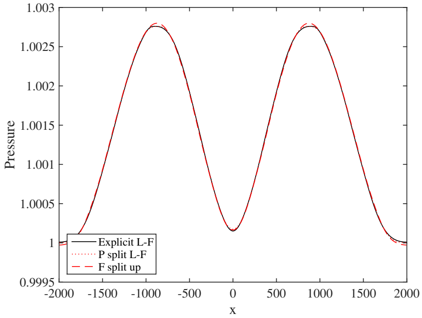
We test two different values: $\epsilon = 1/11$ and $\epsilon = 10^{-3}$.

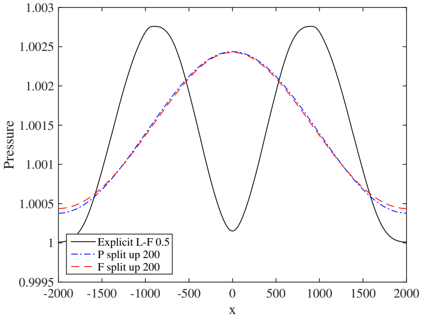


Pressure profile at final time, $\epsilon = 1/11$, $N = 400$ grid points.
CFL = 0.5 - Explicit scheme.
CFL = 3 (material CFL = 0.44) for semi-implicit schemes.

Acoustic waves, small ϵ

Let us now consider $\epsilon = 10^{-3}$.

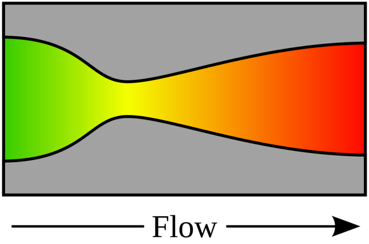




Pressure at final time, $N = 400$
Left: CFL $= 0.5$, right: CFL $= 200$ (material CFL $= 0.39$).

Nozzle flow

Quasi-1D flow model used to describe the acceleration of the propulsion gas in a rocket engine. $A(x)$ represents the cross area.



Laval system

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial m}{\partial x} = -\dfrac{m}{A}\dfrac{\partial A}{\partial x} \\ \dfrac{\partial m}{\partial t} + \dfrac{\partial}{\partial x}(mu + p) = -\dfrac{mu}{A}\dfrac{\partial A}{\partial x} \\ \dfrac{\partial E}{\partial t} + \dfrac{\partial \rho h u}{\partial x} = -\dfrac{\rho h u}{A}\dfrac{\partial A}{\partial x} \end{cases}$$

1D benchmark problem for which an exact solution can be constructed, with variable Mach number.

Semi-implicit first order schemes

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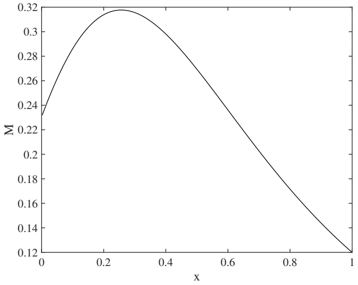
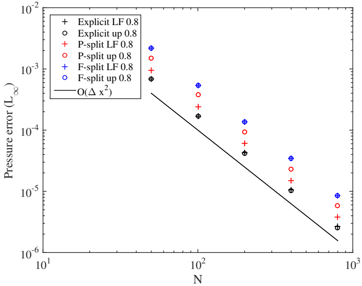
Here we adopt *first order schemes in space and time* which are *well balanced* to second order in space.

Mach $\approx 10^{-1}$



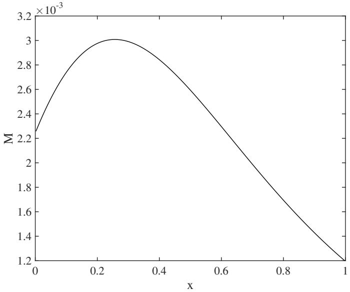
Mach profile.



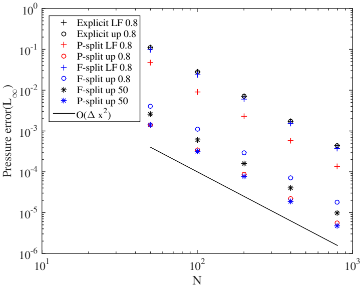
Convergence of the different methods.

CFL = 0.8 for all schemes.
In this regime explicit schemes perform best. F schemes perform worst.

Mach $\approx 10^{-3}$



Mach profile.



Convergence of the different methods.

CFL = 0.8 for explicit, and CFL = 0.8 and 50 for implicit schemes.
Most accurate (and most efficient): P-split upwind CFL 50
Least accurate (end least efficient): Explicit upwind (CFL 0.8)

Transonic flow with shock

To check robustness for large Mach number.

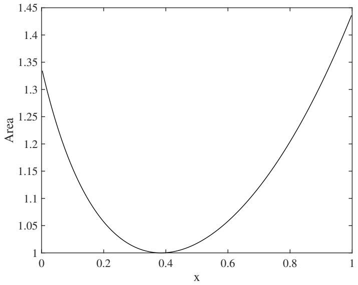


Figure: Geometry of the nozzle for a transonic flow.

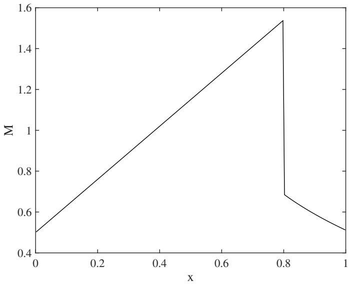
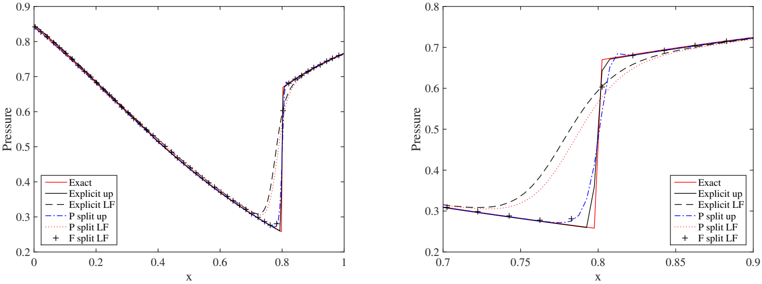


Figure: Mach profile with a shock at x=0.8.

Transonic flow with a shock



Pressure profile and relative zoom around the shock at x=0.8.

Best: explicit upwind. Worst: explicit L-F.
Reasonable: P-split upwind and F-split L-F.

Work in Progress: high order extension

[in collaboration with S. B, J. Qiu, G. Russo.]
What about high order all-Mach number schemes?

first order time semi-discrete scheme

$$\begin{cases} \frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla \cdot (\mathbf{m})^{n+1} = 0 \\ \frac{\mathbf{m}^{n+1}-\mathbf{m}^n}{\Delta t} + \nabla \cdot (\rho u \otimes u)^n + \nabla p(\rho)^{n+1}/\varepsilon^2 = 0, \end{cases}$$

Plug $\mathbf{m}^{n+1}$ into the eq. of the density, one gets[4]

$$\begin{cases} \frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla \cdot (\mathbf{m})^n + \Delta t^2 \nabla^2 : \left(\frac{m^n \otimes m^n}{\rho^n} \right) + \Delta t^2 \frac{1}{\varepsilon^2} \Delta p(\rho^{n+1}) = 0 \\ \frac{(\mathbf{m})^{n+1}-(\mathbf{m})^n}{\Delta t} + \nabla \cdot (\rho u \otimes u)^n + \nabla p(\rho)^{n+1}/\varepsilon^2 = 0, \end{cases}$$

By solving the density equation (it is a non linear elliptic equation) we determine the density at step t^{n+1} and then, the momentum, explicitly.

[4] Di Marco, Loubere, Vignal, 2016

Work in Progress

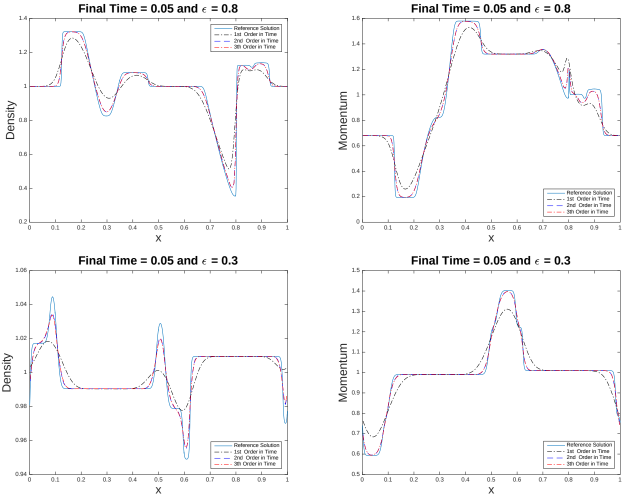
Extension to high order is obtained using the following ingredients:

Upwind derivatives High-order WENO reconstructions for the flux.

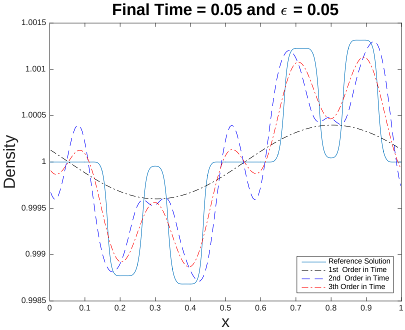
Central derivatives Five points stencils for the Laplacian.

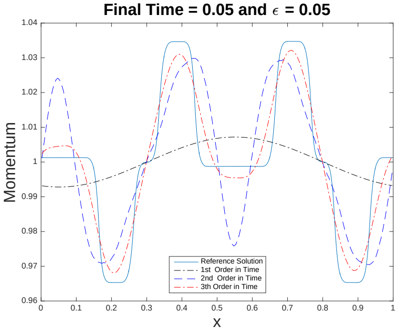
Time integrator Use the high-order SI-IMEX methods introduced in [Boscarino, Filbet. G.R., JSC 2016].

Numerical results for scheme in 1D[5]



[5] Degond and Tang,2011


Final Time = 0.05 and ϵ = 0.05
Density
x
1.0015
1.001
1.0005
1
0.9995
0.999
0.9985
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Reference Solution
1st Order in Time
2nd Order in Time
3th Order in Time


Final Time = 0.05 and ϵ = 0.05
Momentum
x
1.04
1.03
1.02
1.01
1
0.99
0.98
0.97
0.96
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
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Convergence test

IMEX-DIRK(2,3,2), second order in time.

	$\varepsilon = 1$		$\varepsilon = 0.1$		$\varepsilon = 0.0001$	
N	L_1 error	L^1 order	L_1 error	L_1 order	L_1 error	L_1 order
40	3.6427e-02	–	2.8841e-02	—	2.8187e-02	—
80	1.5515e-02	1.1231	1.1944e-02	1.2718	1.1690e-02	1.2697
160	5.1687e-03	1.5858	4.2177e-03	1.5018	4.1803e-03	1.4836
320	1.4425e-03	1.8412	1.2161e-03	1.7942	1.2170e-03	1.7803
640	3.7865e-04	1.9297	3.2851e-04	1.8883	3.2080e-04	1.9235
1280	9.6591e-05	1.9709	8.0737e-05	2.0246	8.0535e-05	1.9940

Convergence test

IMEX-DIRK(3,4,3), third order in time

	$\varepsilon = 1$		$\varepsilon = 0.1$		$\varepsilon = 0.0001$	
N	L_1 error	L^1 order	L_1 error	L_1 order	L_1 error	L_1 order
120	9.3386e-04	–	7.1266e-04	—	6.8998e-04	—
240	1.2187e-04	2.9379	9.1566e-05	2.9603	8.8287e-05	2.9663
480	1.5256e-05	2.9979	1.1396e-05	3.0063	1.0980e-05	3.0073
960	1.8993e-06	3.0059	1.4156e-06	3.0090	1.3639e-06	3.0091

Conclusions and perspectives

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- Future work Generalization to other systems with "all Mach number" problems (gas-fluid, gas-solid).

Thanks for your attention!