# Numerical boundary layers for linear hyperbolic initial-boundary value problems and semigroup estimate

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Joint work with J.-F. Coulombel (Univ. Nantes)

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# A few words about boundary conditions

$$\partial_t u + \mathrm{div}_x f(u) = 0, \quad x \in \Omega \subset \mathbb{R}^d$$
$$u(x,t) = b(x,t), \quad x \in \partial\Omega$$

- Viscous (artificial) parabolic approximation
- Boundary entropy inequalities
- Effective/residual boundary condition :

$$u(x + 0^- \nu(x), t) \in \mathcal{O}(b(x,t)), \quad x \in \partial\Omega$$

- Well-posed problems ($L^1$-contractive semigroup)

*Some references:*

BARDOS, LEROUX & NEDELEC '79
DUBOIS, LEFLOCH '88
GISCLON, SERRE '94
ANDREIANOV, SBIHI '07, '15 : maximal monotone graphs.

# Some related applications

- *Numerical counterpart for 3-points finite volume schemes*
  LEROUX '79 : Convergence for the Godunov and Lax-Friedrichs scheme
  GODLEWSKI, RAVIART '04 : for monotone and *E*-schemes
- *Interfacial coupling in a conservative or nonconservative framework*
  Discontinuous flux conservation laws (large litterature, ...)
  Coupling through admissible trace sets CHALONS, RAVIART & al.
  $L^1$-dissipative germs ANDREIANOV, KARLSEN & RISEBRO
- *Shocks or transitions*
  Singularities in source terms, LAGOUTIERE, SEGUIN, TAKAHASHI, & AGUILLON
  Discrete shock profiles SERRE & al.
  Undercompressive shock profiles (from visco-dispersive approx.) and travelling wave analysis
  Nonclassical shocks and controled entropy dissipation

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# Stability theory for the continuous IBVP

**Non-characteristic linear hyperbolic IBVP**

$$\begin{aligned} &\partial_t u + A\,\partial_x u = F(x,t), \quad && (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ &B\,u(0,t) = g(t), && t \in \mathbb{R}_+ \\ &u(x,0) = f(x), && x \in \mathbb{R}_+ \end{aligned}$$

## Definition (Strong stability for the BVP)

For $f \equiv 0$,

$$\gamma \|e^{-\gamma t} u\|^2_{L^2_t L^2_x} + \|e^{-\gamma t} u_{|x=0}\|^2_{L^2_t} \leq C \left( \frac{1}{\gamma} \|e^{-\gamma t} F\|^2_{L^2_t L^2_x} + \|e^{-\gamma t} g\|^2_{L^2_t} \right).$$

(Fourier-Laplace transform and normal mode analysis, see e.g. [Benzoni-Gavage & Serre])

Strong stability is equivalent to the uniform Kreiss-Lopatinskii condition.
onedimensional case : $\mathbb{R}^N = \mathrm{Ker}\, B \oplus E_+(A)$.

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Strong stability implies **semigroup stability** (multidimensional case) :

- RAUCH '72 for symmetrizable or strictly hyperbolic systems
- AUDIARD '11 for systems with constant multiplicities
- METIVIER '14 for a more general class

Then for all $\gamma > 0$ :

$$e^{-2\gamma T}\|u(\cdot,T)\|^2_{L^2(\mathbb{R}^+)} + \gamma\|e^{-\gamma t}u\|^2_{L^2(\mathbb{R}^+\times[0,T])} + \|e^{-\gamma t}u_{|x=0}\|^2_{L^2([0,T])}$$
$$\leq C\left(\|f\|^2_{L^2(\mathbb{R}^+)} + \frac{1}{\gamma}\|e^{-\gamma t}F\|^2_{L^2(\mathbb{R}^+\times[0,T])} + \|e^{-\gamma t}g\|^2_{L^2([0,T])}\right)$$

# Stability theory for the discrete IBVP

GUSTAFSSON, KREISS & SUNDSTRÖM '72

**Definition (Strong/GKS stability $\ell_t^{2,\gamma}\ell_x^2$)**

$$\frac{\gamma}{1+\gamma\,\Delta t}\sum_{n\geq 0}\Delta t\,\mathrm{e}^{-2\gamma\,n\Delta t}\,\|u^n\|_\Delta^2+\sum_{n\geq 0}\Delta t\,\mathrm{e}^{-2\gamma\,n\Delta t}\,\|u^n\|_\partial^2$$
$$\leq C\left(\frac{1+\gamma\,\Delta t}{\gamma}\sum_{n\geq k}\Delta t\,\mathrm{e}^{-2\gamma\,n\Delta t}\,\|F^n\|_\Delta^2+\sum_{n\geq k}\Delta t\,\mathrm{e}^{-2\gamma\,n\Delta t}\,\|g^n\|_\partial^2\right)$$

Strong stability equivalent to an algebraic condition (UKLC)

**From the discrete Cauchy stability to the strong stability**

- GOLDBERG & TADMOR '81. In the scalar case, considering the Dirichlet boundary condition: the stability for the discrete Cauchy problem implies its strong stability.
- MICHELSON '83. Multidimensional case, dissipative schemes only.

How to include nonzero initial data ?

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# Discrete semigroup stability results

**Cauchy stability + GKS stability ⇒ semigroup stability**

- WU '95. For scalar equations or for one-dimensional systems, for one-step difference schemes.
  *Tool:* by a superposition argument, design auxiliary strictly dissipative boundary conditions, and use the Goldberg-Tadmor result to connect with Dirichlet boundary condition.
- COULOMBEL & GLORIA '11. Extension for systems with several space dimensions and variable coefficients. For one-step difference schemes.
  *Tool:* energy method and another auxiliary dissipative boundary conditions, without using the GKS stability result.
- COULOMBEL '15. Multistep multidimensional systems. + simple roots in the von Neumann Cauchy stability
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# Our setting: multistep MOL schemes

**Scalar one-dimensional transport equation** ($a \neq 0$)

$$\begin{aligned} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0, & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \\ u(x,0) &= f(x), & x \in \mathbb{R}^+ \\ u(0,t) &= 0 \text{ (weak)}, & t \in \mathbb{R}^+ \end{aligned}$$

**Multistep "Method Of Lines" finite difference schemes**

$u_j^n \simeq u(j\Delta x, n\Delta t)$, CFL parameter $\lambda = \Delta t/\Delta x$

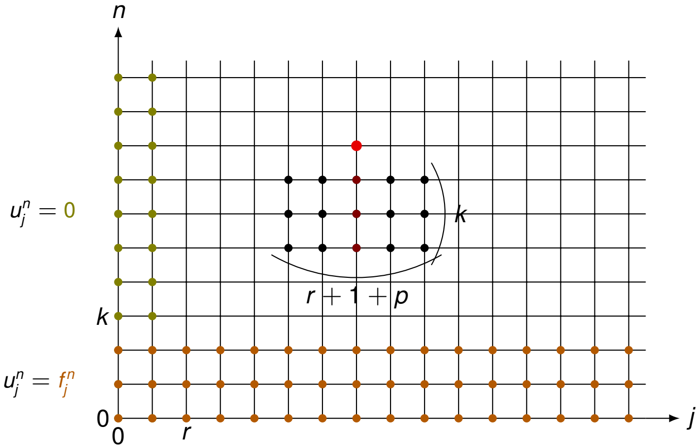
$$\boxed{\sum_{\sigma=0}^{k} \alpha_\sigma u_j^{n+\sigma} + \frac{\Delta t}{\Delta x}\sum_{\sigma=0}^{k-1}\beta_\sigma \sum_{\ell=-r}^{p} a_\ell u_{j+\ell}^{n+\sigma} = 0} \qquad r \leq j \qquad 0 \leq n$$

$$u_j^n = f_j^n := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} f(x - at^n)\, dx \qquad 0 \leq j \qquad 0 \leq n \leq k-1$$

$$u_j^n = 0 \qquad 0 \leq j \leq r-1 \qquad k \leq n$$

# Numerical stencil and notations

$$\sum_{\sigma=0}^{k} \alpha_\sigma u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell u_{j+\ell}^{n+\sigma} = 0, \quad r \leq j, \quad 0 \leq n$$



# Reminder of the pure discrete Cauchy problem

$$\sum_{\sigma=0}^{k} \alpha_\sigma u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell u_{j+\ell}^{n+\sigma} = 0, \quad j \in \mathbb{Z}$$

*Fourier multiplier of the space discretization (von Neumann analysis):*

$$\mathcal{A}(z) = \sum_{\ell=-r}^{p} a_\ell z^\ell, \quad z \neq 0$$

*Linear recurrence relation of the time discretization:*

Characteristic polynomial: $P_\mu(X) = \rho(X) - \mu\sigma(X), \quad \mu \in \mathbb{C}$

with the Dahlquist's generating polynomials:

$$\rho(X) = \sum_{\sigma=0}^{k} \alpha_\sigma X^\sigma, \sigma(X) = \sum_{\sigma=0}^{k-1} \beta_\sigma X^\sigma$$

**Consistency of the numerical scheme**

$$\mathcal{A}(1) = 0, \qquad \mathcal{A}'(1) = a,$$

$$\rho(1) = 0, \qquad \rho'(1) = \sigma(1) \quad (= 1).$$

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# Semigroup stability for the discrete Cauchy problem

The discrete Cauchy problem is supposed to be **semigroup stable** :

$$\exists C > 0, \ \forall \Delta t \in (0,1), \ \forall (f^{\sigma})_{0 \le \sigma \le k-1} \in \left(\ell^2_x(\mathbb{Z})\right)^k$$

$$\sup_{n \ge 0} \|u^n\|_{\ell^2(\mathbb{Z})} \le C \sum_{\sigma=0}^{k-1} \|f^{\sigma}_j\|_{\ell^2(\mathbb{Z})}$$

Power boundedness of the companion matrices in the time recurrence relation → **Stability region:**

$$\mathcal{S} = \left\{ \mu \in \mathbb{C}, \ P_{\mu}(z) = 0 \Rightarrow \begin{pmatrix} |z| < 1, \text{ or} \\ |z| = 1 \text{ and } z \text{ is simple} \end{pmatrix} \right\}.$$

**Common theorem:**

The (semigroup) stability for the Cauchy problem **is equivalent** to:

$$\forall \xi \in \mathbb{R}, \ -\lambda \mathcal{A}(e^{i\xi}) \in \mathcal{S}.$$

[Hairer, Nørsett & Wanner] '93, [Hairer & Wanner] '96

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[HAIRER, NØRSETT & WANNER] '93, [HAIRER & WANNER] '96

# Selected examples (1)

**The everyday ones:**

Time discretization: one-step explicit Euler method

$$P_\mu(X) = X - 1 - \mu.$$

$$\mathcal{S} = \left\{\mu \in \mathbb{C},\ |1+\mu| \leq 1\right\} = D(-1,1).$$

Space discretization:

- upwind $\mathcal{A}(\mathbb{S}^1) = \partial D(1,|a|)$. Stability under CFL condition $|\lambda a| \leq 1$.
- downwind $\mathcal{A}(\mathbb{S}^1) = \partial D(-1,|a|)$. Instability.
- two-points centered $\mathcal{A}(\mathbb{S}^1) = ia[-1,1]$. Instability.

# Selected examples (2)

**A third order explicit scheme : AB3 - 5pts** *(to be continued)*

Time discr.: 3rd order explicit Adams-Bashforth
Space discr.: centered 5pts approximation of the flux term plus a fourth order stabilizing dissipative term

$$u_j^{n+1} = u_j^n - \lambda\left(\frac{23}{12}v_j^n - \frac{16}{12}v_j^{n-1} + \frac{5}{12}v_j^{n-2}\right)$$
$$v_j^n := a\,\frac{-u_{j+2}^n + 8u_{j+1}^n - 8u_{j-1}^n + u_{j-2}^n}{12} - \frac{-u_{j+2}^n + 4u_{j+1}^n - 6u_j^n + 4u_{j-1}^n - u_{j-2}^n}{24}$$

$$P_\mu(X) = X^3 - X^2 - \mu\left(\frac{23}{12}X^2 - \frac{16}{12}X + \frac{5}{12}\right)$$
$$\mathscr{A}(z) = \frac{a}{12}(-z^2 + 8z - 8z^{-1} + z^{-2}) - \frac{1}{24}(-z^2 + 4z - 6 + 4z^{-1} - z^{-2})$$

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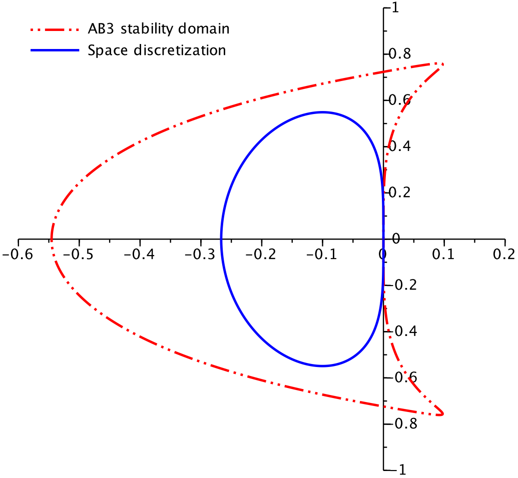
# Selected examples (2)

*Stability assumption:* (CFL parameter $\lambda = 0.4$)



# Discrete semigroup estimate for the IBVP

$$\sum_{\sigma=0}^{k} \alpha_\sigma u_j^{n+\sigma} + \frac{\Delta t}{\Delta x} \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell u_{j+\ell}^{n+\sigma} = 0, \quad u_j^n = 0 \text{ (boundary)}, \quad u_j^n = f_j^n \text{ (initial)}.$$

## Theorem (B. & Coulombel)

*Consider an initial data $f \in H^2(\mathbb{R}^+)$ satisfying the compatibility conditions*

$$\begin{cases} f(0) = 0, & \text{if } a < 0, \\ f(0) = f'(0) = 0, & \text{if } a > 0. \end{cases}$$

*Suppose the above scheme (with zero source data and zero boundary data) to be consistent, Cauchy stable, and "dissipative" (see further).*

$$\sup_{n \leq N_T} \sum_{j \geq 0} \Delta x |u_j^n|^2 \leq C \left( \|f\|^2_{L^2(\mathbb{R}_+)} + \Delta t^{1-3\mu} e^{2T\Delta t^\mu} \|f\|^2_{H^2(\mathbb{R}_+)} \right), \qquad \mu \in [0, 1/3].$$

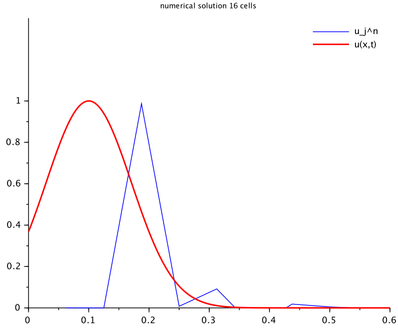
# Numerical experiment

**Test case:** (AB3 - 5pts scheme)

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, \; x \in [0,1], \; t \geq 0,$$

$$u(x,0) = f(x) = e^{-100(x-0.5)^2}, \; x \in [0,1],$$

Solution computed at time $T = 0.4$ with different $\Delta x$



# Strategy

1. Find an expansion $u_j^n = u_{j,n}^{\mathrm{app}} - e_{j,n}$ such that
   - $u_{j,n}^{\mathrm{app}}$ is a sufficiently accurate description of $u_j^n$ including the boundary layer
   - $e_{j,n}$, the residual error terms, solves the discrete IBVP with zero initial data and small boundary terms and small source terms.

$$\sum_{\sigma=0}^{k} \alpha_\sigma \, e_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell \, e_{j+\ell,n+\sigma} = \Delta t \, \varepsilon_{j,n+k}$$

$$e_{j,n} = \eta_{j,n}, \ (0 \leq j \leq r-1)$$

$$e_{j,0} = \cdots = e_{j,k-1} = 0, \ (j \geq 0)$$

where we set

$$\left| \begin{array}{l} \varepsilon_{j,n+k} := \dfrac{1}{\Delta t} \left( \displaystyle\sum_{\sigma=0}^{k} \alpha_\sigma \, u_{j,n+\sigma}^{\mathrm{app}} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell \, u_{j+\ell,n+\sigma}^{\mathrm{app}} \right) \\ \eta_{j,n} := u_{j,n}^{\mathrm{app}} \end{array} \right.$$

2. Goldberg-Tadmor lemma applied to $e_{j,n}$ gives GKS strong estimate
3. + Error estimates for $\varepsilon_{j,n+k}$ and $\eta_{j,n}$ $\Rightarrow$ semigroup estimate for $e_{j,n}$
4. Semigroup estimate on $u_{j,n}^{\mathrm{app}}$

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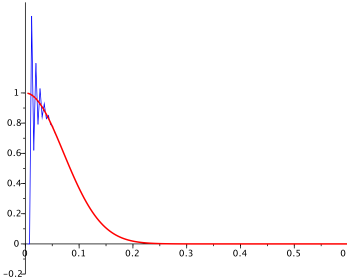
# Heuristics




At the discrete level, two scales $x$ and $\Delta x$.

$$u_j^n \simeq u_{j,n}^{\mathrm{app}} = u^{\mathrm{int}}(x_j, t^n) + u^{\mathrm{bl}}(j, t^n)$$

$$u^{\mathrm{bl}}(j, t^n) = u^{\mathrm{bl},0}(j, t^n) + \Delta x \, u^{\mathrm{bl},1}(j, t^n)$$

- $u^{\mathrm{int}}(x, t)$ corresponds to the smooth part of the solution
- $u^{\mathrm{bl}}(j, t)$ is the sawtoothed pattern localized in the very first cells near the boundary

# ❶ Far from the boundary: $u^{\mathrm{int}}$

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k} \alpha_\sigma\, u^{\mathrm{app}}_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{app}}_{j+\ell,n+\sigma}\right)$$

Fix $x \in \mathbb{R}_+^*$ and let $\Delta t, \Delta x \to 0$.
Then for $x_j \simeq x$, $j \to \infty$ so that $u^{\mathrm{bl}}(j, t^n)$ tends to 0.
Thus

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k} \alpha_\sigma\, u^{\mathrm{int}}_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{int}}_{j+\ell,n+\sigma}\right).$$

$\Leftarrow$ Set $u^{\mathrm{int}}(x,t)$ as the solution of the unbounded domain problem:

$$\partial_t u + a\,\partial_x u = 0, \quad x \in \mathbb{R},\ t \geq 0$$
$$u(x,0) = f(x)\,\mathbb{1}_{\mathbb{R}_+} + 0 \times \mathbb{1}_{\mathbb{R}_-}$$

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$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k}\alpha_\sigma\, u^{\text{int}}_{j,n+\sigma} + \lambda\sum_{\sigma=0}^{k-1}\beta_\sigma\sum_{\ell=-r}^{p} a_\ell\, u^{\text{int}}_{j+\ell,n+\sigma}\right).$$

$\Leftarrow$ Set $u^{\text{int}}(x,t)$ as the solution of the unbounded domain problem:

$$\partial_t u + a\,\partial_x u = 0, \quad x \in \mathbb{R},\ t \geq 0$$
$$u(x,0) = f(x)\mathbb{1}_{\mathbb{R}_+} + 0 \times \mathbb{1}_{\mathbb{R}_-}$$

# ❷ Leading boundary layer profile: $u^{\mathrm{bl},0}$

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k}\alpha_\sigma\, u^{\mathrm{app}}_{j,n+\sigma} + \lambda\sum_{\sigma=0}^{k-1}\beta_\sigma\sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{app}}_{j+\ell,n+\sigma}\right)$$

Fix now $j \in \mathbb{Z}$ and let $\Delta t, \Delta x \to 0$.

$$u^{\mathrm{int}}(x_{j+\ell}, t^{n+\sigma}) = u^{\mathrm{int}}(0, t^n) + O(\Delta x) + O(\Delta t)$$

Suppose moreover some time-regularity in the boundary layer

$$u^{\mathrm{bl}}(j, t^{n+\sigma}) = u^{\mathrm{bl}}(j, t^n) + O(\Delta t),$$

then

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta x}\left(\sum_{\sigma=0}^{k-1}\beta_\sigma\right)\sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{bl},0}(j+\ell, t^{n+k}) + O(1).$$

$\Leftarrow$ Set $(u^{\mathrm{bl},0}(j,t))_j$ a solution of $\sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{bl},0}(j+\ell,t) = 0$, together with the boundary conditions $u^{\mathrm{bl},0}(j,t) = -u^{\mathrm{int}}(0,t), \quad 0 \le j \le r-1$, and the limiting behavior $\lim_{j\to\infty} u^{\mathrm{bl},0}(j,t) = 0$.

# ❷ Leading boundary layer profile: $u^{\mathrm{bl},0}$

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# ❷ Leading boundary layer profile: $u^{\mathrm{bl},0}$

## Definition

Being given $u \in \mathbb{R}$, a sequence $(v_j)_{j\in\mathbb{N}}$ is said to be a stable boundary layer profile associated with $u$ if:

1. $v_0 = \cdots = v_{r-1} = -u$,
2. $\sum_{\ell=-r}^{p} a_\ell v_{j+\ell+r} = 0$ for all $j \geq 0$,
3. $\lim_{j\to\infty} v_j = 0$.

Denote $\mathcal{C}_{\mathrm{num}}$ the set of all $u$ such that a stable boundary layer exists.

Identify the set $\mathcal{C}_{\mathrm{num}}$ ?
Being given $u \in \mathcal{C}_{\mathrm{num}}$, is there a unique associated stable boundary layer profile ?

- DUBOIS & LEFLOCH '88 - admissible entropy boundary data
- GISCLON & SERRE '97 - residual boundary conditions for the Godunov scheme
- CHAINAIS-HILLAIRET & GRENIER '01 - conservative schemes

# Technical "dissipativity" assumption

Stable boundary layers are obtained by considering roots of $\mathcal{A}$, with $|z| < 1$.

**Assumption (H)**

$z = 1$ is the unique root of $\mathcal{A}$ on $\mathbb{S}^1$

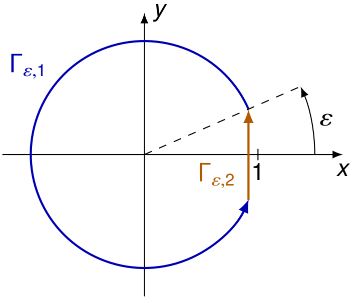
**Lemma**

*Under the Cauchy stability assumption and the above assumption (H),* $\mathcal{A}(z) = 0$ *admits exactly R roots (with multiplicity) in* $\left\{z \in \mathbb{C}\,,\, 0 < |z| < 1\right\}$

$$\text{where } R = \begin{cases} r, & \text{if } a < 0\,, \\ r - 1, & \text{if } a > 0\,. \end{cases}$$

# Proof of the Lemma

$$\frac{1}{2\,i\,\pi}\int_{\Gamma}\frac{\mathcal{A}'(z)}{\mathcal{A}(z)}\,\mathrm{d}z = \#\{\text{zeros}\} - \#\{\text{poles}\}\,,$$




- 0 is pole of order $r$
- 1 is zero of order 1 : $\mathcal{A}(1) = 0$, $\mathcal{A}'(1) = a \neq 0$
- does not vanish on $\Gamma_{\epsilon,1}$, therefore[1] $\mathcal{A}(z) \notin \mathbb{R}_-^*$ / use $\log_-$
- $a\mathcal{A}(z) \notin \mathbb{R}_+$ for $z \in \Gamma_{\epsilon,2}$ ($\epsilon$ being sufficiently small):
  - case $a < 0$ : $\mathcal{A}(z) \notin \mathbb{R}_-$ for $z \in \Gamma_{\epsilon,2}$ / use $\log_-$ : $R = r$
  - case $a > 0$ : $\mathcal{A}(z) \notin \mathbb{R}_+$ for $z \in \Gamma_{\epsilon,2}$ / use $\log_+$ : $R = r - 1$.

[1] The stability region $\mathcal{S}$ contains no positive real number

# Example for selected schemes

**Assumption (H)**

$z = 1$ is the unique root of $\mathcal{A}$ on $\mathbb{S}^1$

★ Explicit Euler time discretization: $P_\mu(X) = X - 1 - \mu$

$$\mathcal{A}(e^{i\eta}) = 0 \Leftrightarrow 1 - \lambda\mathcal{A}(e^{i\eta}) = 1$$

- Any dissipative scheme satifies (H):

  $\exists c > 0,\ \exists m \in \mathbb{N}^*,\ \forall |\eta| \leq \pi,\ |1 - \lambda\mathcal{A}(e^{i\eta})| \leq 1 - c\eta^{2m}.$

- Some other usual non-dissipative schemes also satisfy (H).
  The Lax-Friedrichs scheme:

  $u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\lambda a}{2}(u_{j+1}^n - u_{j-1}^n),$

  $1 - \lambda\mathcal{A}(e^{i\eta}) = \cos\eta - i\lambda a \sin\eta.$

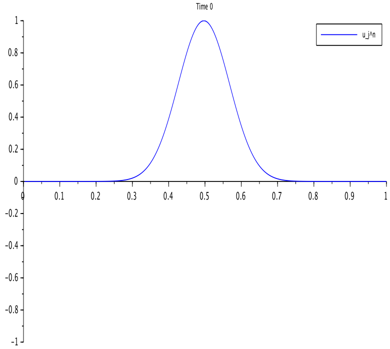
★ The AB3 - 5pts scheme satisfies also (H)

$$\Re\mathcal{A}(e^{i\eta}) = \frac{2}{3}\sin^4\left(\frac{\eta}{2}\right),$$

# Example for selected schemes (2)

★ The leap frog scheme as a (well-known) counterexample ($N = 300$)

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} + a\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0,\ 1 \leq j \leq N-1, \qquad u_0^n = u_N^n = 0.$$




$$P_\mu(X) = \frac{1}{2}(X^2 - 1) - \mu X$$

$$\mathcal{A}(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)$$

$$\mathcal{A}(1) = \mathcal{A}(-1) = 0$$

bounded oscillating pattern:
$u_j^n = (-1)^{j+n}$

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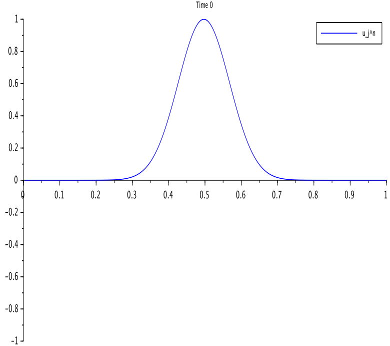


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bounded oscillating pattern:
$u_j^n = (-1)^{j+n}$

# Back to ❷ Leading boundary layer profile: $u^{\mathrm{bl},0}$

Consequently, comparing the number of (independant) generators for the boundary layer to the number of Dirichlet boundary datas :

## Lemma

- *if* $a > 0$*, then* $\mathcal{C}_{\mathrm{num}} = \{0\}$ *and the unique boundary layer profile associated with* $u = 0$ *is the zero sequence*
- *if* $a < 0$*, then* $\mathcal{C}_{\mathrm{num}} = \mathbb{R}$ *and for any* $u \in \mathbb{R}$ *there is a unique stable boundary layer profile* $(v_j)_{j\in\mathbb{N}}$ *associated with* $u$*, that decreases exponentially fast at infinity.*

$$v_j = u\, w_j\,, \quad j \geq 0\,,$$

*where* $(w_j)_{j\in\mathbb{N}}$ *denotes the (canonical) boundary layer profile associated with* $u = 1$*.*

$$u^{\mathrm{bl},0}_{j,n} = u^{\mathrm{int}}(0, t^n) w_j.$$

# ❸ First boundary layer corrector: $u^{\mathrm{bl},1}$

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k} \alpha_\sigma\, u^{\mathrm{app}}_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{app}}_{j+\ell,n+\sigma}\right)$$

Remainder terms (up to every previous approximation) are

$$\varepsilon_{j,n+k} \simeq \frac{1}{\Delta t} \sum_{\sigma=0}^{k} \alpha_\sigma\, u^{\mathrm{bl},0}(j, t^{n+\sigma}) + \left(\sum_{\sigma=0}^{k-1} \beta_\sigma\right) \sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{bl},1}(j+\ell, t^n)\,.$$

To be solved :

$$w_j + \sum_{\ell=-r}^{p} a_\ell\, \widetilde{w}_{j+\ell} = 0,\ j \geq r\,,$$

$$\widetilde{w}_0 = \cdots = \widetilde{w}_{r-1} = 0\,, \qquad \lim_{j \to \infty} \widetilde{w}_j = 0\,.$$

**Lemma**

*In the case $a < 0$, there exists a unique solution $(\widetilde{w}_j)_{j \in \mathbb{N}}$ and this solution decays exponentially fast at infinity.*

# ❸ First boundary layer corrector: $u^{\mathrm{bl},1}$

$$\varepsilon_{j,n+k} := \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k} \alpha_\sigma\, u^{\mathrm{app}}_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{app}}_{j+\ell,n+\sigma}\right)$$

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Remainder terms (up to every previous approximation) are

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To be solved :

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# Numerical experiment around the boundary layer expansion

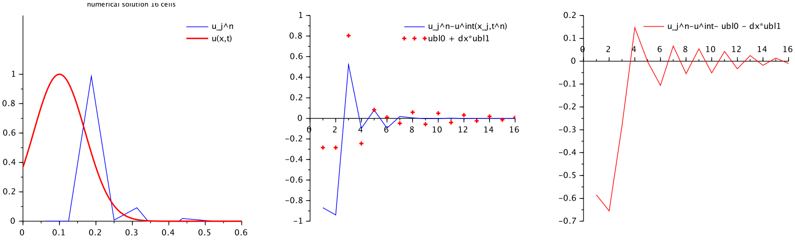
## Comparisons

**Test case:** (AB3 - 5pts scheme)

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0, \; x \in [0,1], \; t \geq 0,$$
$$u(x,0) = f(x) = e^{-100(x-0.5)^2}, \; x \in [0,1],$$

Root of $\mathcal{A}$ in $\{z \in \mathbb{C}, \; 0 < |z| < 1\}$: $z_1 \simeq -0.6595$ and $z_2 \simeq 0.0809$

$$u_j^n \simeq u_{j,n}^{\text{app}} := u_{j,n}^{\text{int}} + u_{j,n}^{\text{bl},0} + \Delta x \, u_{j,n}^{\text{bl},1}$$

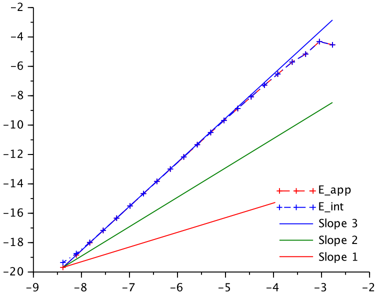



Solution computed at time $T = 0.4$ with different $\Delta x$

# Numerical experiment around the boundary layer expansion

## Rate of convergence $\ell^2$

$$E_2^{\mathrm{int}} := \left(\sum_{j=0}^{N} \Delta x \left|u_j^n - u^{\mathrm{int}}(x_j, t^n)\right|^2\right)^{1/2}, \quad E_2^{\mathrm{app}} := \left(\sum_{j=0}^{N} \Delta x \left|u_j^n - u^{\mathrm{app}}(x_j, t^n)\right|^2\right)^{1/2}.$$




At time T=0.125 : no significant boundary layer at $x = 0$.

$$E_2^{\mathrm{int}} = O(\Delta x^3), \quad E_2^{\mathrm{app}} = O(\Delta x^3)$$

# Numerical experiment around the boundary layer expansion

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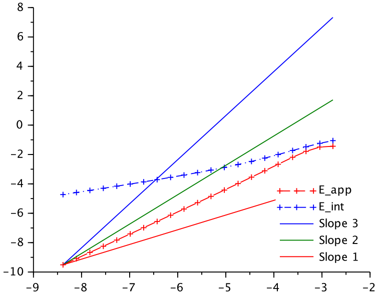


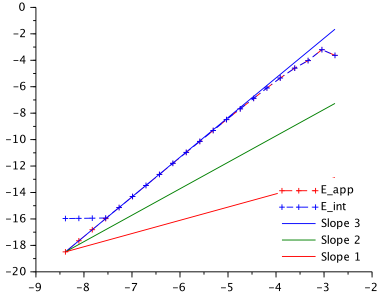

At time T=0.4 : a boundary layer.

$$E_2^{\mathrm{int}} = O(\Delta x^{1/2}), \quad E_2^{\mathrm{app}} = O(\Delta x^{3/2})$$

# Numerical experiment around the boundary layer expansion

## Rate of convergence $\ell^\infty$

$$E_\infty^{\mathrm{int}} := \max_{0\le j\le N} |u_j^n - u^{\mathrm{int}}(x_j, t^n)|, \quad E_\infty^{\mathrm{app}} := \max_{0\le j\le N} |u_j^n - u^{\mathrm{app}}(x_j, t^n)|.$$




At time T=0.125 : no significant boundary layer at $x = 0$.

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## Rate of convergence $\ell^\infty$

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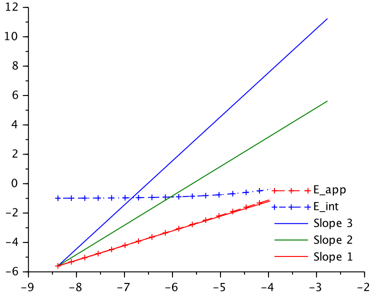


At time T=0.4 : a boundary layer.

$$E_\infty^{\mathrm{int}} = O(\Delta x^0), \quad E_\infty^{\mathrm{app}} = O(\Delta x^1)$$

# "GKS estimate" for the error terms

Recall we set
$$\varepsilon_{j,n+k} := \frac{1}{\Delta t}\left(\sum_{\sigma=0}^{k} \alpha_\sigma\, u^{\mathrm{app}}_{j,n+\sigma} + \lambda \sum_{\sigma=0}^{k-1} \beta_\sigma \sum_{\ell=-r}^{p} a_\ell\, u^{\mathrm{app}}_{j+\ell,n+\sigma}\right)$$
$$\eta_{j,n} := u^{\mathrm{app}}_{j,n}$$

Then, $\exists C > 0,\ \forall \Delta t \in (0,1],\ \forall \gamma > 0,\ \forall f \in H^2_0(\mathbb{R}^+)$ :

$$\sum_{n\geq k}\sum_{j\geq r} \Delta t\, \Delta x\, e^{-2\gamma n \Delta t}\, |\varepsilon_{j,n}|^2 \leq C\left(1+\frac{1}{\gamma}\right)\Delta t^2\, \|f\|^2_{H^2(\mathbb{R}^+)},$$

$$\sum_{n\geq k}\sum_{j=0}^{r-1} \Delta t\, e^{-2\gamma n \Delta t}\, |\eta_{j,n}|^2 \leq C\,\Delta t^2\, \|f\|^2_{H^1(\mathbb{R}^+)}.$$

*Some ingredients:*

- Consistency of the interior scheme
- Compatibility condition : homogeneous Dirichlet/initial data
- Exponential decrease in space of the boundary layer

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# Semigroup estimate for the error term

GOLDBERG AND TADMOR '81: For homogeneous Dirichlet conditions and under the discrete Cauchy stability assumption, one has the GKS estimate

$$\exists C > 0,\ \forall \Delta t \in (0, 1],\ \forall \gamma > 0 :$$

$$\frac{\gamma}{1+\gamma\,\Delta t} \sum_{n\geq 0} \sum_{j\geq 0} \Delta t\,\Delta x\,\mathrm{e}^{-2\,n\gamma\,\Delta t}\,|e_j^n|^2 + \sum_{n\geq 0} \sum_{j=0}^{r+p-1} \Delta t\,\mathrm{e}^{-2\,n\gamma\,\Delta t}\,|e_j^n|^2$$

$$\leq C \left( \frac{1+\gamma\,\Delta t}{\gamma} \sum_{n\geq k} \sum_{j\geq r} \Delta t\,\Delta x\,\mathrm{e}^{-2\,n\gamma\,\Delta t}\,|\varepsilon_j^n|^2 + \sum_{n\geq k} \sum_{j=0}^{r-1} \Delta t\,\mathrm{e}^{-2\,n\gamma\,\Delta t}\,|\eta_j^n|^2 \right)$$

$$\leq C\,\Delta t^2\,\|f\|^2_{H^2(\mathbb{R}^+)} \left( \frac{1+\gamma\Delta t}{\gamma} \left( 1 + \frac{1}{\gamma} \right) + 1 \right),$$

To make it readable, choose $\gamma = 1$, we easily get:

$$\sup_{n\geq 0} \left( \mathrm{e}^{-2\,t^n} \sum_{j\geq 0} \Delta x\,|e_j^n|^2 \right) \leq C\,\Delta t\,\|f\|^2_{H^2(\mathbb{R}^+)}$$

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# Semigroup estimate for the numerical solution

From the previous semigroup estimate for the error terms:

$$\sup_{n\geq 0}\left(\mathrm{e}^{-2\,t^n}\sum_{j\geq 0}\Delta x\,|e_j^n|^2\right)\leq C\,\Delta t\,\|f\|^2_{H^2(\mathbb{R}^+)}\,,$$

and from an direct semigroup estimate concerning the boundary layer expansion:

$$\sum_{j\geq 0}\Delta x\,|u_{j,n}^{\mathrm{app}}|^2\leq C\,\|f\|^2_{L^2(\mathbb{R}^+)}$$

Finally, using a triangular inequality ($u_j^n = u_{j,n}^{\mathrm{app}} - e_j^n$), we get :

$$\sum_{j\geq 0}\Delta x\,|u_j^n|^2\leq C\left(\|f\|^2_{L^2(\mathbb{R}^+)}+\Delta t\,e^{2t^n}\,\|f\|^2_{H^2(\mathbb{R}^+)}\right).$$

**Remark:**
without the corrector $u_{j,n}^{\mathrm{bl},1}$, the last estimate would lose the $\Delta t$ factor.

# Semigroup estimate for the numerical solution

From the previous semigroup estimate for the error terms:

$$\sup_{n\geq 0}\left(\mathrm{e}^{-2\,t^n}\sum_{j\geq 0}\Delta x\,|e_j^n|^2\right)\leq C\,\Delta t\,\|f\|^2_{H^2(\mathbb{R}^+)},$$

and from an direct semigroup estimate concerning the boundary layer expansion:

$$\sum_{j\geq 0}\Delta x\,|u_{j,n}^{\mathrm{app}}|^2\leq C\,\|f\|^2_{L^2(\mathbb{R}^+)}$$

Finally, using a triangular inequality ($u_j^n = u_{j,n}^{\mathrm{app}} - e_j^n$), we get :

$$\sum_{j\geq 0}\Delta x\,|u_j^n|^2\leq C\left(\|f\|^2_{L^2(\mathbb{R}^+)}+\Delta t\,e^{2t^n}\,\|f\|^2_{H^2(\mathbb{R}^+)}\right).$$

**Remark:**
without the corrector $u_{j,n}^{\mathrm{bl},1}$, the last estimate would lose the $\Delta t$ factor.

# Conclusions and perspectives

- Main result: close to optimal semigroup stability estimate for the discrete IBVP, compatible in the limit with the continuous one:

$$\sup_{t\geq 0} \|u(\cdot,t)\|^2_{L^2(\mathbb{R}^+)} \leq K \|u(\cdot,0)\|^2_{L^2(\mathbb{R}^+)}$$

- The two-scale asymptotic boundary layer expansion allows the treatment of MOL multistep schemes.
- The discrete boundary layer structure is not directly related to the equivalent equation of the scheme.

★ Up to now, the approach is restricted to Dirichlet boundary conditions, for which the strong GKS stability estimate is known to hold under the discrete Cauchy stability.

★ Explore higher order boundary layer expansions (up to the order of accuracy of the numerical scheme), and initial layers as well.

★ Weaken the (H) assumption on the spatial discretization.

★ Export the tool to the multidimensional situations.

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# Some references

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