Non-compactness of initial data sets in high dimensions.

Seminar on Mathematical General Relativity
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Scalar-field theory in General Relativity

Let $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, be a Lorentzian manifold, $\Psi \in C^\infty(\mathcal{M}^{n+1})$ a scalar-field and $V \in C^\infty(\mathbb{R})$ a potential.

$(\mathcal{M}^{n+1}, h, \Psi)$ is said to be a a space-time if it satisfies the following Einstein equations:

$$\begin{cases} Ric(h)_{ij} - \frac{1}{2} R(h) h_{ij} = \nabla_i \Psi \nabla_j \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{ij}, \\ \Box_h \Psi = \frac{dV}{d\Psi}. \end{cases} \tag{E}$$

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Relevant physical cases:

- Vacuum case with no cosmological constant: $\Psi \equiv 0$, $V \equiv 0$.
- Vacuum case with positive cosmological constant: $\Psi \equiv 0$, $V \equiv \Lambda > 0$.
- Klein-Gordon fields: $V(\Psi) = \frac{1}{2} m \Psi^2$, $m > 0$.

The Evolution Problem

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Notion of initial data sets on M^n:

Theorem (Choquet-Bruhat '52, Choquet-Bruhat-Geroch '69)

$(M^n \times \mathbb{R}, h, \Psi)$ solves (E) if and only if $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ solves in M^n the contraint system:

$$\begin{cases} R(\tilde{g}) + tr_{\tilde{g}}\tilde{K}^2 - ||\tilde{K}||_{\tilde{g}}^2 = \tilde{\pi}^2 + |\tilde{\nabla}\tilde{\psi}|_{\tilde{g}}^2 + 2V(\tilde{\psi}) \ , \\ \tilde{\nabla}(\mathrm{tr}_{\tilde{g}}\tilde{K}) - div_{\tilde{g}}K = -\tilde{\pi}\tilde{\nabla}\tilde{\psi} \ , \end{cases} \qquad (C)$$

In particular: any solution $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ of (C) evolves into a solution of the Einstein equations. A solution $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ is therefore called an initial data set.

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Here we have let:

- $\tilde{g} = h_{|M^n}$ and $\tilde{\nabla}$ is the Levi-Civita connection for $\tilde{g}$ in M^n,
- $\tilde{K}$: second fundamental form of the embedding $M^n \subset M^n \times \mathbb{R}$,
- $\tilde{\psi} = \Psi_{|M^n}$ and $\tilde{\pi} = (N \cdot \Psi)_{|M^n}$. N is the future-directed unit normal to M^n.

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System (C) has $n(n+1) + 2$ unknowns $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ for $n + 1$ equations.

The conformal method (Lichnerowicz, Choquet-Bruhat, York)

Goal: produce solution of the constraint equations:

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Conformal parametrization: look for the unknown initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ as:

$$\left(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}\right) = \left(u^{\frac{4}{n-2}} g, \frac{\tau}{n} u^{\frac{4}{n-2}} g + u^{-2}\left(\sigma + \mathcal{L}_g W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right), \qquad (*)$$

where $u \in C^\infty(M)$, $u > 0$, $W \in T^*M$ and $\mathcal{L}_g W$ is the conformal Killing operator.

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where $u \in C^\infty(M)$, $u > 0$, $W \in T^*M$ and $\mathcal{L}_g W$ is the conformal Killing operator.

These data depend on $n+1$ parameters (u, W) and on given physics data $(\psi, \pi, \tau, \sigma, V)$ where:

- V is the potential of the scalar-field,
- ψ, π are scalar-field data,
- τ is a mean curvature,
- σ, is a $(2,0)$-symmetric tensor field with $\mathrm{tr}_g \sigma = 0$ and $\mathrm{div}_g \sigma = 0$ ("TT tensor").

The Einstein-Lichnerowicz constraint system

The conformal parametrization solves the original constraint equations if and only if (u, W) solve the Einstein-Lichnerowicz constraint system:

$$\begin{cases} \triangle_g u + hu = fu^{2^*-1} + \dfrac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = u^{2^*} X + Y. \end{cases} \tag{CC}$$

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Here: $2^* = \frac{2n}{n-2}$, (u, W) are smooth, $u > 0$. Also $\triangle_g = -\mathrm{div}_g(\nabla \cdot)$, $\triangle_g = -\mathrm{div}_g(\nabla \cdot)$. $\mathcal{L}_g W$ is the conformal Killing derivative:

$$\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n} div_g W \cdot g_{ij},$$

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In the physical case, the coefficients $(h, f, \pi, \sigma, X, Y)$ depend on the choice of the given physics data $(\psi, \pi, \tau, \sigma, V)$ of the conformal method.

Our goal: understand the blow-up behavior of the solutions of (CC).

Setting of our problem:

In the following: for us, M will alway be compact without boundary. The coefficients $(h, f, \pi, X, Y, \sigma)$ will satisfy the assumptions of the focusing case:

$$f > 0, \qquad \triangle_g + h \quad \text{coercive}, \qquad \text{and} \qquad \pi \not\equiv 0.$$

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In the physical case, the coefficients are related to the physics data by:

$$h = \frac{n-2}{4(n-1)}\left(S_g - |\nabla \psi|_g^2\right),$$
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In the following we will investigate the system for general focusing coefficients $(h, f, \pi, X, Y, \sigma)$, not only the physical ones.

Criticality of the system and defects of compactness

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The critical nonlinearity u^{2^*-1}, with the “mean” focusing sign $f > 0$, implies that concentration phenomena (or blow-up) are likely to occur.

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Model example: standard bubbles. For $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, $n \geq 3$:

$$B_{\lambda,x_0}(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-x_0|^2}{n(n-2)}} \right)^{\frac{n-2}{2}} , \quad \triangle_\xi B_{\lambda,x_0} = B_{\lambda,x_0}^{2^*-1} \quad \text{in } \mathbb{R}^n, \quad \|B_{\lambda,x_0}\|_{L^{2^*}} = K_n.$$

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Similar explosive phenomena in $C^0(M)$ are obtained for critical nonlinear elliptic equations or systems (Druet-Hebey '04, Robert-Vétois '14, Pistoia-Vaira '15, Vétois-Thizy '16...).

Perturbations of the coefficients increase the chance of appearance of defects of compactness.
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Natural Question: when do these blow-up phenomena occur for the EL system?

The notion of Stability for the Einstein-Lichnerowicz constraint system

Definition

Let $(h, f, \pi, X, Y, \sigma) \in C^2(M)$. The Einstein-Lichnerowicz system is said to be ***stable*** *if, for any sequence $(h_k, f_k, \pi_k, X_k, Y_k, \sigma_k)_k$ converging to $(h, f, \pi, X, Y, \sigma)$ in $C^2(M)$ and for any sequence $(u_k, W_k)_k$ of solutions of:*

$$\begin{cases} \triangle_g u_k + h_k u_k = f_k {u_k}^{2^*-1} + \dfrac{\pi_k^2 + |\sigma_k + \mathcal{L}_g W_k|_g^2}{{u_k}^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_k = {u_k}^{2^*} X_k + Y_k, \end{cases}$$

there exists a solution (u, W) of (CC) such that $(u_k, W_k) \to (u, W)$ in $C^2(M)$ (up to a subsequence and up to elements in the kernel of $\mathcal{L}_g$).

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The stability of an equation/system yields structural informations. Instability is a failure of uniform (in the choice of the coefficients) a priori bounds for solutions.

Stability results

Stability holds (on a locally conformally flat manifold) under the following conditions:

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- If $3 \leq n \leq 5$ (Druet-P. '14, $n = 3$, P. '15)
- If $n \geq 6$ and ∇f and X have no common zero in M. Or, if they do, provided at these zeroes there holds:

$$h < \frac{n-2}{4(n-1)} S_g - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_g f}{f}. \tag{0.1}$$

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What about the sharpness of these conditions in high dimensions: can blow-up phenomena happen in high dimensions?

Main result: instability examples in high dimensions

Theorem (Non-compactness in high dimensions $n \geq 6$, P., '16)

Let (M, g) be a closed Riemannian manifold of dimension $n \geq 6$, such that $\overrightarrow{\triangle}_g$ has no kernel. There exist coefficients $(h, f, \pi, \sigma, X, Y)$ of class C^2, satisfying the assumptions of the focusing case and $X \not\equiv 0$ such that the Einstein-Lichnerowicz system:

$$\begin{cases} \triangle_g u + hu = fu^{2^*-1} + \dfrac{|\mathcal{L}_g W + \sigma|_g^2 + \pi^2}{u^{2^*+1}} \\ \overrightarrow{\triangle}_g W = u^{2^*} X + Y \end{cases} \tag{0.2}$$

possesses a blowing-up sequence of solutions $(u_k, W_k)_k$, that is: $\|u_k\|_{L^\infty(M)} \to +\infty$ and $\|\mathcal{L}_g W_k\|_{L^\infty(M)} \to +\infty$ as $k \to +\infty$. Here the u_k are positive, have one concentration point and have a non-zero limit profile.

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In particular: these coefficients $(h, f, \pi, \sigma, X, Y)$ satisfy

$$h \geq \frac{n-2}{4(n-1)} S_g - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_g f}{f} \text{ somewhere.}$$

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possesses a blowing-up sequence of solutions $(u_k, W_k)_k$, that is: $\|u_k\|_{L^\infty(M)} \to +\infty$ and $\|\mathcal{L}_g W_k\|_{L^\infty(M)} \to +\infty$ as $k \to +\infty$. Here the u_k are positive, have one concentration point and have a non-zero limit profile.

In particular: these coefficients $(h, f, \pi, \sigma, X, Y)$ satisfy

$$h \geq \frac{n-2}{4(n-1)} S_g - \frac{(n-2)(n-4)}{8(n-1)} \frac{\triangle_g f}{f} \text{ somewhere.}$$

Surprising consequence: the Einstein-Lichnerowicz system has an infinite number of (blowing-up) solutions in high dimensions!

Two dual approaches in the blow-up analysis of critical elliptic equations

1) The *a priori* approach. It is the one used to prove stability results.

Answers the following question: given an (arbitrary) blowing-up sequence of solutions of EL, what can I say about it? It gives informations about: the pointwise blow-up behavior of sequences of solutions, the localisation of concentration points, the mutual interactions between different defects of compactness,...

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In P., '15 it is for instance proven that any blowing-up sequence (u_k, W_k) of the E-L system satisfies:

$$u_k = B_k + o(B_k) \text{ in } C^0$$

in the neighbourhood of a concentration point, where B_k is a given bubbling profile modeled on the standard bubble. And, as a consequence, that at a concentration point x_0 there holds:

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Approach developed by: Li, Zhu, Druet, Schoen, Marques, Zhang, Khuri, Hebey, Robert.

Two dual approaches in the blow-up analysis of critical elliptic equations II

2) The Lyapounov-Schmidt approach, or H^1-constructive approach: used to construct blowing-up sequences of solutions under suitable assumptions on the coefficients.

Idea: look for solutions as:

$$u_{t,\xi,k} = B_{t,\xi,k} + u_0 + \varphi_{t,\xi,k},$$

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Example: to solve $\triangle_g u + hu = u^{2^*-1}$, find $u_{t,\xi,k}$ critical point of the energy. Reduces to find (t,ξ) critical point of:

$$(t,\xi) \mapsto \frac{1}{2}\int_M |\nabla B_{t,\xi,k}|^2 + hB_{t,\xi,k}^2 dv_g - \frac{1}{2^*}\int_M B_{t,\xi,k}^{2^*} dv_g$$

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Developed by Wei, Rey, Del Pino, Pacard (over the last 15 years)

Proof: A constructive approach *via* a glueing method

We hope to find blowing-up solutions of the Einstein-Lichnerowicz system:

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The system is strongly coupled ($X \not\equiv 0$) and the vector equation is supercritical in the natural energy space $H^1(M)$: the system is non-variational and ill-posed in $H^1(M)$. The system therefore exhibits a double (super-)criticality that cannot be handled with standard constructive energy methods.

Solution: a C^0 constructive approach that relies on the a priori analysis

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$$|\varphi_{t,\xi,k}| \le \varepsilon_k \left(B_{t,\xi,k} + u_0\right), \tag{0.3}$$

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Our solution depends on $(n+1)$ parameters (t,ξ) – just like the standard bubbling profiles.

Goal: find, for every k, a value $(t_k,\xi_k)_k$ of the parameters and a suitable remainder $\varphi_{t_k,\xi_k,k}$ (small in $C^0(M)$) for which $u_{t_k,\xi_k,k}$ is indeed a solution!

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By the choice of φ and X, we now have pointwise estimates on this W_k that blows-up:

$$|\mathcal{L}_g W_{t,\xi,k}| \sim \frac{\mu_k^{\frac{n-1}{2}}}{\left(\mu_k^2 + d_{g_\xi}(\xi,x)^2\right)^{\frac{n-1}{2}}} \text{ close to } \xi.$$

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Problem: it blows up too fast to plug it into the scalar equation and perform a usual ping-pong method!

Sketch of the proof II: Semi-decoupling

Step 1: Construct a solution of:

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Done via a nonlinear fixed-point method in H^1 in the orthogonal of the kernel of the linearized equation at $B_{t,\xi,k}$ (spanned by the $Z_{j,k}$). Here $\mathcal{L}_g W_{t,\xi,k}$ is a coefficient.

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Goal: Get an (almost) solution of the system if $\psi = \varphi$.

Sketch of the proof III: Fixed-point in C^0 1

Step 2: The goal is now to fix-point $\varphi \mapsto \psi$, in the set of C^0 functions satisfying:

$$|\varphi| \leq \varepsilon_k \left(B_{t,\xi,k} + u_0\right). \tag{0.4}$$

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This is again done in three steps:

Step a): Extend the *a priori* asymptotic techniques of the C^0-theory of Druet-Hebey-Robert to this scalar equation. Possible here since the red term comes with explicit (and suitable) pointwise bounds.

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This shows that

$$\psi = o \left(B_{t,\xi,k} + u_0 \right) \quad \text{in } C^0(M).$$

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$$+ \|h - c_n S_g\|_{L^\infty} d_g(\xi, x)^2 + d_g(\xi, x)^4 \mathbb{1}_{nlcf}\Bigg] B_{t,\xi,k}(x) + \left(\frac{\mu_k}{\mu_k + d_g(\xi, x)}\right)^2.$$

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We also prove that these estimates are uniform in t, ξ and φ.

Step c): Choose a suitable ε_k (according to the red term). And then show that $\varphi \mapsto \psi$ is a contraction. Relies on the second-order estimates.

Sketch of the proof IV: Concluding argument

Step 3: At the end of Step 2, after point-fixing the remainders, we have a solution $(u_{t,\xi,k}, W_{t,\xi,k})$ of:

$$\begin{cases} \triangle_g u + hu = fu^{2^*-1} + \dfrac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} + \displaystyle\sum_{j=0}^{n} \lambda_k^j(t,\xi)\,(\triangle_g + h)\, Z_{j,k,t,\xi} \ , \\ \overrightarrow{\triangle}_g W = u^{2^*} X + Y, \end{cases}$$

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To conclude: use the second-order estimates on $\varphi_{t,\xi,k}$ to obtain an asymptotic expansion of the $\lambda_{k,j}(t,\xi)$ in $C^0_{loc}(\mathbb{R}^{n+1})$ as $k \to +\infty$. And we are left to annihilate $(n+1)$ functions from $\mathbb{R}^{n+1}$ to $\mathbb{R}$.

Thank you for your attention.

Bonus: Explicit expressions of h and X.

The explicit expressions of h, f and X are the following:

$$f(x) \approx f_0$$

$$h(x) = \frac{n-2}{4(n-1)} S_g(x) + \sum_{k \geq 1} \tau_k H\left(\frac{1}{\beta_k}\left(\exp_{\xi_0}\right)^{-1}(x)\right),$$

$$X(x) = X_0(x) + \sum_{k \geq 1} \mu_k^{\frac{n-1}{2}} Z\left(\left(\exp_{\xi_0}\right)^{-1}(x)\right),$$

where τ_k depends on μ_k and on the dimension and if (M, g) is locally conformally flat or not. Also, $\mu_k << \beta_k << 1$ is another scale parameter.

The function H has a strict local maximum at 0 and $|Z(0)|_\xi > 0$.

Bonus: Explicit expressions of h and X.

The explicit expressions of h, f and X are the following:

$$
\begin{aligned}
f(x) &\approx f_0 \\
h(x) &= \frac{n-2}{4(n-1)} S_g(x) + \sum_{k \geq 1} \tau_k H\left(\frac{1}{\beta_k}\left(\exp_{\xi_0}\right)^{-1}(x)\right), \\
X(x) &= X_0(x) + \sum_{k \geq 1} \mu_k^{\frac{n-1}{2}} Z\left(\left(\exp_{\xi_0}\right)^{-1}(x)\right),
\end{aligned}
$$

where τ_k depends on μ_k and on the dimension and if (M, g) is locally conformally flat or not. Also, $\mu_k << \beta_k << 1$ is another scale parameter.

The function H has a strict local maximum at 0 and $|Z(0)|_\xi > 0$.

We did not jut play around with the values of the parameters so that everything fits well in the end: the relations between the parameters are rigid and are given by the a priori pointwise stability analysis.