Precise asymptotics for the wave equation on stationary, asymptotically flat spacetimes

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Outline

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1. *Warm-up*: late-time behaviour of the wave equation on Minkowski

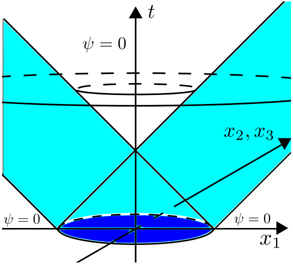
Consider the wave equation on 3+1-dimensional Minkowski space:

$$-\partial_t^2\psi + \sum_{i=1}^{3} \partial_i^2\psi = 0,$$

with initial data

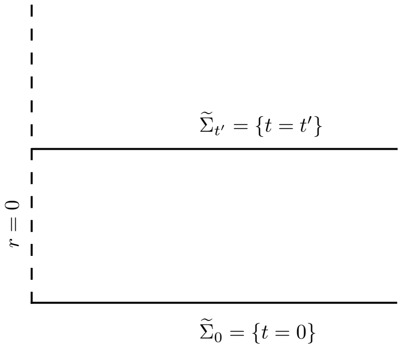
$$\psi(0,\cdot) \in C_0^\infty(\mathbb{R}^3),$$
$$\partial_t\psi(0,\cdot) \in C_0^\infty(\mathbb{R}^3).$$

The strong Huygens principle: if ψ is initially supported in dark blue ball, then ψ vanishes outside the *wave zone* (depicted below in light blue). This follows from Kirchoff's formula for $\psi(t,x)$.



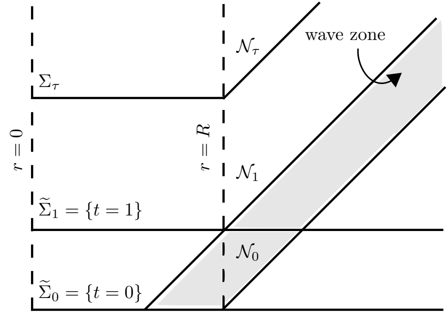
We obtain *boundedness* of solutions ψ to the wave equation by using energy conservation:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3}(\partial_t\psi)^2(t,x)+\sum_{i=1}^{3}(\partial_i\psi)^2(t,x)\,dx\right)=0.$$



We want to capture <u>*decay*</u> of solutions ψ to the wave equation by using energy conservation with respect to asymptotically null hypersurfaces:

$$\frac{d}{d\tau}E[\tau] \leq 0.$$



"Energy at time $\tau = 1$" equals "energy at time $\tau = 0$" plus "energy radiated to infinity between $\tau = 0$ and $\tau = 1$".

2. Statement of main result

Let $\mathcal{M}$ be a $3+1$-dimensional manifold equipped with a metric g of signature $(-,+,+,+)$ (a "spacetime") that is:

1. stationary,
2. asymptotically flat.

Consider the initial value problem for the corresponding geometric wave equation

$$\Box_g \psi = \frac{1}{\sqrt{-\det g}} \sum_{\alpha,\beta} \partial_\alpha \left(\sqrt{-\det g} (g^{-1})^{\alpha\beta} \partial_\beta \psi \right) = 0$$

with smooth, compactly supported Cauchy data.

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with smooth, compactly supported Cauchy data. Assume moreover:

3. *energy boundedness* and *integrated local energy decay* for the wave equation.

For this talk, let us fix $(\mathcal{M}, g)$ to be the *domain of outer communications of a Schwarzschild black hole*: $\mathcal{M} = \mathbb{R} \times [2M, \infty) \times \mathbb{S}^2$ and

$$g = -(1 - 2M/r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

with $M > 0$.

Theorem (Y. Angelopoulos–S. Aretakis–D.G. '16).
Let ψ be a solution to $\Box_g \psi = 0$ on Schwarzschild, arising from smooth, compactly supported initial data. Then we can estimate

$$\left|\psi(\tau, r, \theta, \varphi) - P \cdot \frac{1}{(1+\tau)^3}\right| \lesssim D(1+\tau)^{-3-\epsilon} \quad \text{in } \{r \leq R\},$$

$$\left|\psi(u, v, \theta, \varphi) - \frac{1}{2} P \cdot \frac{1}{u^2 v}\left(1 + \frac{u}{v}\right)\right| \lesssim D(1+v)^{-1}(1+u)^{-2-\epsilon} \quad \text{in } \{r \geq R\},$$

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4. The above polynomial decay rates were first derived heuristically by Price '72 for fixed modes ψ_ℓ and are sometimes referred to as "Price's law".

3. Motivations and previous work

Motivations

General relativity is a theory of gravity that describes the interaction of energy and matter with spacetime curvature according to the Einstein equations. In vacuum these are given by:

$$\mathrm{Ric}_{\alpha\beta}(g) = 0.$$

In *harmonic coordinates* these reduce to:

$$\Box_g g_{\alpha\beta} = N_{\alpha\beta}(g, \partial g).$$

The wave equation on a spacetime solution to the Einstein equations is the simplest linear toy model for the dynamics of spacetime perturbations in the context of the Cauchy problem for the Einstein equations:

1. Robust, quantitative, uniform *upper bound* time-decay estimates are needed for proving global existence and uniqueness of nonlinear wave equations and for proving global stability of spacetime solutions to the Einstein equations,

   See the proof of global stability of Minkowski by Christodoulou–Klainerman '91.

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2. Quantitative uniform *lower bound* time-decay estimates *outside* black holes are relevant for understanding the geometry *inside* black holes.

Previous work

Recently there has been a lot of activity in proving boundedness and decay estimates for solutions to the wave equation on domains of outer communications of black holes. See for example work by:

Andersson, Angelopoulos, Aretakis, Bachelot, Blue, Chodosh, Civin, Dafermos, Donninger, Finster, Ionescu, Kamran, Kay, Klainerman, Luk, Marzuola, Metcalfe, Moschidis, Oh, Rodnianski, Schlag, Schlue, Shlapentokh-Rothman, Smoller, Soffer, Sterbenz, Stogin, Tataru, Tohaneanu, Wald, Whiting, Yau,...

In the $\Lambda > 0$ (cosmological) and $\Lambda < 0$ (anti de Sitter) settings, see work by:

Bony, Dafermos, Dold, Dunn, Dyatlov, Gannot, Häfner, Hintz, Holzegel, Melrose, Rodnianski, Sà Barreto, Shao, Smulevici, Vasy, Warnick,...

In addition: a plethora of heuristics and numerics by the physics community.

Previous sharp upper bound results

Dafermos–Rodnianski '03 proved a $\tau^{-3+\epsilon}$ upper bound for spherically symmetric ψ in dynamic black hole solutions to the Einstein-scalar field equations (and a τ^{-2} upper bound for $r \cdot \psi$ using L^1 estimates).

Tataru '09 proved a τ^{-3} upper bound for ψ (and a τ^{-2} upper bound for $r \cdot \psi$) in a general class of stationary spacetimes containing Schwarzschild via Fourier methods and resolvent estimates (see also a later extension to certain non-stationary spacetimes by Metcalfe–Tataru–Tohaneanu '11).

Donninger–Schlag–Soffer '09 obtained $\tau^{-2\ell-2}$ upper bound on Schwarzschild for $\psi_{\geq \ell}$ with $\ell \geq 1$.

Previous lower bound results

Luk–Oh '15 showed that

$$\int_1^{\infty} v^7 (\partial_v \psi)^2 \, \sin\theta d\theta d\varphi dv \Big|_{r=2M}$$

blows up for generic smooth, compactly supported initial data on Schwarzschild (and more generally, sub-extremal Reissner–Nordström).

4. **The Klainerman commutating vector field method**

Klainerman's vector field method in a nutshell

KLAINERMAN '85:

Use energy conservation:

$$\frac{d}{dt}\left(\int_{\mathbb{R}^3}(\partial_t\psi)^2(t,x)+\sum_{i=1}^{3}(\partial_i\psi)^2(t,x)\,dx\right)=0,$$

together with:

$$\Box_g(Z\psi)=0$$

if $\Box_g\psi=0$, with Z a vector field generating a Poincaré symmetry in Minkowski or the conformal scaling symmetry.

Obtain decay by using that some of the Z have *growing weights* in t and r.

Commuting with Z and applying a **Klainerman–Sobolev inequality** in Minkowski, gives:

$$|\psi| \leq C(t+r+1)^{-1}(t-r+1)^{-\frac{1}{2}},$$

where the constant $C \geq 0$ depends on weighted initial L^2 norms.

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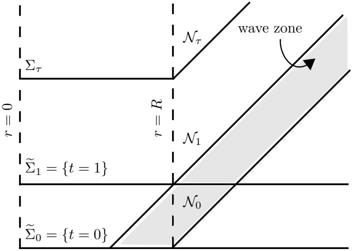
4. The above ideas have been adapted to obtain $t^{-3/2}$ decay in Schwarzschild in Luk '09 and slowly rotating Kerr in Luk '10 inside a region $\{r \leq \frac{t}{2}\}$.

4. **The Dafermos–Rodnianski hierarchy of r^p-weighted estimates**

Dafermos–Rodnianski '09:

Philosophy: capture *quantitatively* the "radiation to infinity" corresponding to a foliation by asymptotically null hypersurfaces. We will sketch the method in Minkowski.



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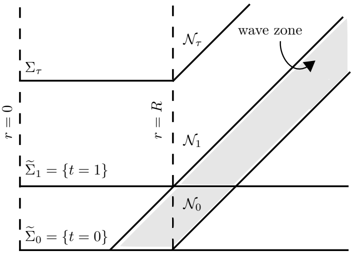

Split the spacetime slab $\tau_1 \leq \tau \leq \tau_2$ into two regions: $\{r < R\}$ and $\{r \geq R\}$. First, let us *assume* integrated local energy decay:

$$\int_{\tau_1}^{\tau_2} \left[\int_{\Sigma_\tau \cap \{r \leq R\}} (\partial\psi)^2 d\mu \right] d\tau \leq C \int_{\Sigma_{\tau_1}} (\partial\psi)^2 + (\partial\partial_t\psi)^2 \, d\mu.$$

Allow for loss of derivatives due to "trapping of null geodesics".

Write $v = t + r$ and $u = t - r$. Integrate by parts the spacetime integral of $r^p \partial_v(r\psi) \cdot \Box_g(r\psi)$ with $p = 1, 2$ to obtain the following (schematic) *hierarchy* of estimates in $\{r \geq R\}$:

$$\int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} (\partial_v(r\psi))^2 + r^2 |\nabla\!\!\!/ \psi|^2 \, d\omega dv \right] du \leq C \int_{\mathcal{N}_{\tau_1}} r \cdot (\partial_v(r\psi))^2 d\omega dv + \ldots,$$

$$\int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} r \cdot (\partial_v(r\psi))^2 d\omega dv \right] du \leq C \int_{\mathcal{N}_{\tau_1}} r^2 \cdot (\partial_v(r\psi))^2 d\omega dv + \ldots.$$

Apply mean-value theorem on dyadic intervals $[\tau_j, \tau_{j+1}]$, $j \in \mathbb{N}$, to obtain:

$$\int_{\mathcal{N}_{\tau'_{j+1}}} (\partial_v(r\psi))^2 + r^2 |\nabla\!\!\!/ \psi|^2 \, d\omega dv \leq C(1 + \tau'_j)^{-2}$$

Together with local integrated energy decay and energy boundedness: for all $\tau \geq 0$

$$\int_{\Sigma_\tau} (\partial\psi)^2 d\mu_\tau \leq C(1 + \tau)^{-2}.$$

By standard Sobolev inequalities, obtain:

$$|\psi| \leq C(1+r)^{-1}(1+\tau)^{-\frac{1}{2}},$$
$$|\psi| \leq C(1+r)^{-\frac{1}{2}}(1+\tau)^{-1}.$$

Commute $\Box_g$ with ∂_v or $r \cdot \partial_v$ (Schlue '10, Moschidis '15) to obtain moreover:

$$|\psi| \leq C(1+\tau)^{-\frac{3}{2}},$$

cf. commuting with the scaling vector field $S = u \cdot \partial_u + v \cdot \partial_v$ and considering conformal energy in Klainerman's method.

5. Conserved radiative quantities and higher-order hierarchies

Strategy: *Extend* the Dafermos–Rodnianski hierarchy in order to apply the mean-value theorem *more* times and consequently get *better* energy decay rates.

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Main idea: Commute $\Box_g$ with $r^2 \cdot \partial_v$ and obtain the following hierarchy for $\Phi = r^2\partial_v(r\psi)$:

$$\int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} (\partial_v\Phi)^2 d\omega dv\right] d\tau \leq C \int_{\mathcal{N}_{\tau_1}} r \cdot (\partial_v\Phi)^2 d\omega dv + \ldots,$$

$$\int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} r \cdot (\partial_v\Phi)^2 d\omega dv\right] d\tau \leq C \int_{\mathcal{N}_{\tau_1}} r^2 \cdot (\partial_v\Phi)^2 d\omega dv + \ldots.$$

Now observe that a Hardy inequality gives:

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} r^2 \cdot (\partial_v(r\psi))^2 d\omega dv\right] d\tau &= \int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} r^{-2} \cdot (r^2\partial_v(r\psi))^2 d\omega dv\right] d\tau \\ &\leq C \int_{\tau_1}^{\tau_2} \left[\int_{\mathcal{N}_\tau} (\partial_v(r^2\partial_v(r\psi))^2 d\omega dv\right] d\tau.\end{aligned}$$

Some remarks

1. **Caveat**: above method works only for $\psi - \int_{\mathbb{S}^2} \psi \, d\omega$.

   More generally, if we decompose $\psi = \sum_{\ell=0}^{\infty} \psi_\ell$, where $\not{\Delta}_{\mathbb{S}^2} \psi_\ell = -\ell(\ell+1)\psi_\ell$, we can keep on commuting with $r^2 \cdot \partial_v$, provided we **subtract** $\psi_0, \psi_1, \ldots, \psi_\ell$, with $\ell > 0$ suitable large.

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2. For each fixed ψ_ℓ, with $\ell \geq 0$, we obtain a *sharp* number of hierarchies if we commute with vector fields of the form $w(r) \cdot \partial_v$, where the functions $w(r) = r^2(1 + O(r^{-1}))$ have to be chosen carefully.

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4. Observe that the commutator vector field $r^2 \cdot \partial_v$ resembles the conformal Killing vector field $K = u^2 \cdot \partial_u + v^2 \cdot \partial_v$, which produces the conformal energy when used as a multiplier.

Aside: extremal Reissner–Nordström

In extremal Reissner–Nordström black hole spacetimes,

$$g = -(1 - M/r)^2 dv^2 + 2dvdr + r^2(d\theta^2 + \sin\theta^2 d\phi^2),$$

there exists a conformal transformation that maps $\{r \geq R\}$ to a neighbourhood of the event horizon at $r = M$ (Couch–Torrence '84).

The above estimates in $\{r \geq R\}$ can therefore be adapted to obtain precise asymptotics for the wave equation on extremal Reissner–Nordström which are fundamental for studying the black hole interior D.G. '15 and are relevant for non-linear problems (*work in progress*). See also previous work by Aretakis '10.

Conserved quantities

The quantity $\lim_{v\to\infty}\left(\int_{\mathbb{S}^2} r^2\partial_v(r\psi)\, d\omega\right)(u,v)$ is conserved in u, i.e. it is equal to a constant I_0 determined from initial data for ψ.

For each fixed ℓ, the $\ell+1$-th order commuted quantities
$r^2\partial_v[(w_1(r)\partial_v)\cdot(w_2(r)\partial_v)\cdot\ldots(w_\ell(r)\partial_v)(r\psi_\ell)]$ are *conserved* along null infinity.

These conserved quantities were first discovered by Newman–Penrose '68 and are called *Newman–Penrose constants*.

The Newman–Penrose constants not only suggest which commutator vector fields to consider in the r-weighted estimates, but their conservation can also be viewed as the *source* of lower bounds and polynomial tails appearing in the late-time asymptotics for the wave equation!

Theorem (Y. Angelopoulos–S. Aretakis–D.G. '16).
Let ψ be a solution to $\Box_g \psi = 0$ on Schwarzschild, arising from smooth, compactly supported initial data. Then we can estimate

$$\left|\psi(\tau, r, \theta, \varphi) - 8I_0^{(1)} \cdot \frac{1}{(1+\tau)^3}\right| \lesssim_\epsilon D(1+\tau)^{-3-\epsilon} \quad \text{in } \{r \leq R\},$$

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where D is a constant depending on initial data for ψ **and** $I_0^{(1)}$ **is the** $\ell = 0$ **Newman–Penrose constant of the time integral** $\psi^{(1)}$**, which is a smooth function satisfying** $\Box_g \psi^{(1)} = 0$ **and** $\partial_\tau \psi^{(1)} = \psi$.

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1. $I_0^{(1)}$ can be determined explicitly from **initial data** for ψ.
2. The constant $I_0^{(1)}$ is generically **non-zero** in Schwarzschild, but it is always **zero** in Minkowski.

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$$\left|\psi(u, v, \theta, \varphi) - 4 I_0^{(1)} \cdot \frac{1}{u^2 v}\left(1 + \frac{u}{v}\right)\right| \lesssim_\epsilon D(1+v)^{-1}(1+u)^{-2-\epsilon} \quad \text{in } \{r \ge R\},$$

where D is a constant depending on initial data for ψ **and $I_0^{(1)}$ is the $\ell = 0$ Newman–Penrose constant of the time integral $\psi^{(1)}$, which is a smooth function satisfying $\Box_g \psi^{(1)} = 0$ and $\partial_\tau \psi^{(1)} = \psi$.**

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2. The constant $I_0^{(1)}$ is generically **non-zero** in Schwarzschild, but it is always **zero** in Minkowski.

Theorem (Y. Angelopoulos–S. Aretakis–D.G. '16).
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3. Can obtain similar estimates for $\partial_\tau^k \psi$ where the decay rate in τ or u increases by k.

Thank you!