The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge

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The celebrated Kerr family of spacetimes comprise a 2-parameter family of black hole solutions to the Einstein vacuum equations:

$$\mathrm{Ric}[g] = 0. \tag{1}$$

The physical reality of such objects requires a positive resolution to the conjectured stability of their exterior regions:

Conjecture

The Kerr exterior family is stable as a family of solutions to (1).

A more precise formulation, in analogy with the monumental work of Christodoulou–Klainerman establishing the stability of the Minkowski space, is in the context of general relativity as an initial value problem:

> *"Do initially small perturbations of initial data for a Kerr exterior solution evolve, under the Einstein vacuum equations, to a nearby member of the Kerr family?"*

To resolve this conjecture one must first address the issue of gauge freedom associated to the diffeomorphism invariance of general relativity.

Particular success in studying the Einstein equations has come from imposing a wave gauge:

In vacuum:

- the fundamental local existence result of Choquet-Bruhat (1952)
- the pioneering nonlinear stability of the Minkowski space, Lindblad–Rodnianski (2010)

Einstein-scalar field, Einstein–Maxwell, Einstein–Klein–Gordon, Einstein–Vlasov:

- the nonlinear stability of the Minkowski space – Lindblad–Rodnianski (2010), Speck (2014), LeFloch–Ma (2015), Fajman–Joudioux–Smulevici and Lindblad–Taylor (2017)

Motivated by this success, one has the aim of resolving the conjecture in the affirmative by imposing a generalised wave gauge:

- natural generalisation of the wave gauge to the situation where one is perturbing about a spacetime with non-trivial curvature
- the linearisation of the Einstein equations in this gauge about a non-trivial background solution exhibit an amenable structure

In this talk we shall make precise the following theorem which establishes the quantitative linear stability of the Schwarzschild exterior family:

Theorem (J.)

All smooth and asymptotically flat solutions to the system of equations that result from linearising the Einstein vacuum equations, as expressed in a generalised wave gauge, about a fixed member of the Schwarzschild exterior family remain uniformly bounded on the Schwarzschild exterior and in fact decay at an inverse polynomial rate to a member of the linearised Kerr family after the addition of a (explicit) dynamically determined residual pure gauge solution.

Related results

Generalised wave gauge:

- the nonlinear stability of the Kerr–De Sitter and Kerr–Newman–De Sitter family of black holes with small rotation parameter, Hintz–Vasy (2016) and Hintz (2016)
- the nonlinear stability of the Minkowski space, Hintz–Vasy (2017)

The linear stability of the Schwarzschild exterior family:

- by imposing a double–null gauge, Dafermos–Holzegel–Rodnianski (2016)

There is also a result due to Hung–Keller–Wang (2017) which employed a Chandresekhar gauge.

To resolve via a generalised wave gauge first requires upgrading this linear theory from the Schwarzschild subfamily to the full Kerr family.

Nevertheless, Dafermos, Holzegel and Rodnianski formulated a restricted nonlinear stability conjecture regarding the Schwarzschild exterior family (see end of talk) for which the rate of dispersion we obtain is in principle sufficient to resolve by imposing a generalised wave gauge.

Remarkably, a proof of this conjecture in the class of axially symmetric and polarised perturbations has very recently been announced by Klainerman–Szeftel over a series of three papers, the first of which can be found on the arXiv.

I. The equations of linearised gravity around Schwarzschild

The Einstein equations in a generalised wave gauge

Let $(\mathcal{M}, \boldsymbol{g})$ and $(\mathcal{M}, \overline{\boldsymbol{g}})$ be $3+1$ globally hyperbolic Lorentzian manifolds with $\boldsymbol{f}$ a smooth vector field on $\mathcal{M}$ and define the connection tensor of $\boldsymbol{g}$ and $\overline{\boldsymbol{g}}$

$$\left(\boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}}\right)^{a}_{\ bc} := \frac{1}{2}\boldsymbol{g}^{ad}\left(2\overline{\boldsymbol{\nabla}}_{(b}\boldsymbol{g}_{c)d} - \overline{\boldsymbol{\nabla}}_{d}\boldsymbol{g}_{bc}\right)$$

where $\overline{\boldsymbol{\nabla}}$ is the Levi-Civita connection of $\overline{\boldsymbol{g}}$.

Then we say that $\boldsymbol{g}$ is in a generalised $\underline{\boldsymbol{f}}$-wave gauge with respect to $\overline{\boldsymbol{g}}$ iff

$$\boldsymbol{g}^{-1} \cdot \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} = \boldsymbol{f}.$$

In this gauge the Einstein vacuum equations $\mathbf{Ric}[\boldsymbol{g}] = 0$ have the <u>schematic</u> description

$$\left(\boldsymbol{g}^{-1} \cdot \overline{\boldsymbol{\nabla}}^{2}\right)\boldsymbol{g} + \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} \cdot \overline{\boldsymbol{\nabla}}\boldsymbol{g} + \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} \cdot \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} + \overline{\mathbf{Riem}} \cdot \boldsymbol{g} = \mathcal{L}_{\boldsymbol{f}}\overline{\boldsymbol{g}} \cdot \boldsymbol{g}$$

with $\overline{\mathbf{Riem}}$ the Riemann tensor of $\overline{\boldsymbol{g}}$.

Remarks

Local well-posedness:

- for a given smooth $\boldsymbol{f}$ and $\overline{\boldsymbol{g}}$ the generalised wave gauge is always locally well-posed
- for a given smooth $\boldsymbol{f}$ and $\overline{\boldsymbol{g}}$ the Einstein vacuum equations in a generalised wave gauge are locally well-posed

The wave gauge:

- setting $(\boldsymbol{f}, \boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}) = (0, \mathbb{R}^4, \eta)$ and choosing a globally inertial coordinate system yields the wave gauge

The conjectured stability of the Kerr exterior family:

- set $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ to be any fixed member of the subextremal exterior Kerr family!

This strategy was employed successfully by Hintz and Vasy for the Kerr–De Sitter family.

The Schwarzschild exterior solution

Let $M > 0$.

Then the Schwarzschild exterior family $(\mathcal{M}, g_M)$ of solutions to the Einstein vacuum equations are defined as the Lorentzian manifolds with boundary written in the regular Schwarzschild-star coordinate system as

$$\mathcal{M} = \mathbb{R} \times [2M, \infty) \times S^2,$$
$$g_M = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^{*2} + \frac{4M}{r} \mathrm{d}t^* \,\mathrm{d}r + \left(1 + \frac{2M}{r}\right) \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\varphi^2\right)$$

- the causal vector field $T = \partial_{t^*}$ defines a time orientation and is manifestly Killing
- the boundary $\mathcal{H}^+$ is a null hypersurface termed the *event horizon*
- the level sets Σ_{t^*} of t^* are asymptotically flat Cauchy hypersurfaces

The equations of linearised gravity

We consider the Einstein vacuum equations as expressed in a generalised wave gauge $\boldsymbol{f}$-wave gauge with respect to a fixed g_M on the manifold $\mathcal{M}$:

$$\left(\boldsymbol{g}^{-1} \cdot \nabla_M^2\right)\boldsymbol{g} + \boldsymbol{C}_{\boldsymbol{g},g_M} \cdot \nabla_M \boldsymbol{g} + \boldsymbol{C}_{\boldsymbol{g},g_M} \cdot \boldsymbol{C}_{\boldsymbol{g},g_M} + \mathrm{Riem}_M \cdot \boldsymbol{g} = \mathcal{L}_{\boldsymbol{f}} g_M \cdot \boldsymbol{g}, \quad (2)$$

$$\boldsymbol{g}^{-1} \cdot \boldsymbol{C}_{\boldsymbol{g},g_M} = \boldsymbol{f}. \quad (3)$$

To formally linearise these equations about the solution $(g_M, 0)$ we consider a smooth 1-parameter family $\big(\boldsymbol{g}(\epsilon), \boldsymbol{f}(\epsilon)\big)$ of solutions to (2)-(3) on $\mathcal{M}$ with $\big(\boldsymbol{g}(0), \boldsymbol{f}(0)\big) = \big(g_M, 0\big)$ with the formal power series expansion

$$\boldsymbol{g}(\epsilon) = g_M + \epsilon \cdot \overset{(1)}{g} + o\big(\epsilon^2\big), \qquad \boldsymbol{f}(\epsilon) = \epsilon \cdot \overset{(1)}{f} + o\big(\epsilon^2\big)$$

which we insert into (2)-(3) and discard higher order terms in ϵ.

Note in particular that since $C_{g_M,g_M} = \nabla_M g_M = 0$ the terms that are first order in the derivatives of $\boldsymbol{g}$ in (2) will vanish under linearisation!

Proceeding as outlined one arrives at the equations of linearised gravity

$$\Box \overset{(1)}{g} - 2\mathrm{Riem} \cdot \overset{(1)}{g} = \mathcal{L}_{\overset{(1)}{f}} g_M,$$
$$\mathrm{div} \overset{(1)}{g} - \frac{1}{2}\, \mathrm{d}\mathrm{tr}_{g_M} \overset{(1)}{g} = \overset{(1)}{f}$$

with $\Box$, div, d the wave operator, divergence and exterior derivative on $(\mathcal{M}, g_M)$.

These equations are well-posed with a smooth solution $\mathscr{S} := \left(\overset{(1)}{g}, \overset{(1)}{f}\right)$ arising uniquely from freely prescribed smooth seed data $\mathscr{D}$.

The latter prescribes $\overset{(1)}{f}$ and determines smooth Cauchy data for $\overset{(1)}{g}$ on an initial hypersurface Σ satisfying constraints.

Q. (towards linear stability) Do sufficiently regular solutions $\mathscr{S}$ to the equations of linearised gravity decay, with a sufficient rate, to the future?

Special solutions

The first such class arise from expressing the exterior Kerr family in a generalised $\underline{\boldsymbol{f}}_K$-wave gauge with respect to g_M and linearising about g_M.

For instance, linearising the Schwarzschild exterior family in the mass parameter yields the 1-parameter family of spherically symmetric solutions

$$\overset{(1)}{g}_{\mathfrak{m},0} = -\frac{1}{4}\frac{1}{1-\frac{2M}{r}}\frac{\mathfrak{m}}{M}\left(\frac{2M}{r}\,\mathrm{d}t^*\otimes\mathrm{d}r-\mathrm{d}r\otimes\mathrm{d}r\right)+\frac{r^2}{2}\frac{\mathfrak{m}}{M}\mathring{g}, \quad \overset{(1)}{f}_{\mathfrak{m},0} = -\frac{1}{2}\frac{1}{r}\frac{\mathfrak{m}}{1-\frac{2M}{r}}\left(\mathrm{d}t^*-\mathrm{d}r\right)$$

with $\mathfrak{m}\in\mathbb{R}$ and $\mathring{g}$ the unit metric on the round sphere.

We denote this family of linearised Kerr solutions by $\mathscr{K}$. In light of their stationarity we must instead ask the question

Q. (towards linear stability) Do sufficiently regular solutions $\mathscr{S}$ to the equations of linearised gravity decay, with a sufficient rate, to a member of the linearised Kerr family $\mathscr{K}$?

The correct $\mathscr{K}$ can in fact be identified from the seed data $\mathscr{D}$ alone!

The second class of special solutions arise from residual gauge freedom.

Indeed, given smooth vector fields v and f satisfying $\Box v = f$ then the smooth 1-parameter family of Lorentzian metrics $\phi_\epsilon^* g_M$, with ϕ_ϵ the smooth 1-parameter family of diffeomorphisms generated by v, are in a generalised f-wave gauge with respect to g_M to first order in ϵ.

This yields the class of pure gauge solutions $\mathscr{G}$ to the equations of linearised gravity:

$$\overset{(1)}{g}_{\mathscr{G}} = \mathcal{L}_v g_M, \qquad \overset{(1)}{f}_{\mathscr{G}} = f \qquad \text{with} \qquad \Box v = f.$$

In light of diffeomorphism invariance we must allow for the possibility

Q. (towards linear stability) Do sufficiently regular solutions $\mathscr{S}$ to the equations of linearised gravity decay, when expressed in a suitable gauge, to a member of the linearised Kerr family $\mathscr{K}$?

To determine the correct $\mathscr{G}$ requires analysing the structure of the equations of linearised gravity.

II. An effective scalarisation of the equations of linearised gravity

We consider two scalar quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ constructed from $\overset{(1)}{g}$ which vanish for all linearised Kerr and pure gauge solutions.

Remarkably, as was first discovered by Regge–Wheeler and Zerilli, the equations of linearised gravity force the decoupling of these *gauge-invariant* quantities into the scalar wave equations described by the celebrated Regge–Wheeler and Zerilli equations respectively:

$$\Box\left(r^{-1}\overset{(1)}{\Phi}\right) = -\frac{4}{r^2}\frac{\mu}{r}\overset{(1)}{\Phi},$$

$$\Box\left(r^{-1}\overset{(1)}{\Psi}\right) = -\frac{4}{r^2}\frac{\mu}{r}\overset{(1)}{\Psi} + \frac{6}{r^2}\frac{\mu}{r}(2-3\mu)\not{\mathcal{C}}^{[1]}\overset{(1)}{\Psi} + \frac{18}{r}\frac{\mu}{r}\frac{\mu}{r}(1-\mu)\not{\mathcal{C}}^{[2]}\overset{(1)}{\Psi}.$$

Here, $\mu = \frac{2M}{r}$ and $\not{\mathcal{C}}^{[p]}$ is the inverse of the elliptic operator $\mathring{\Delta} + 2 - \frac{6M}{r}$ applied p-times with $\mathring{\Delta}$ the Laplacian on the unit round sphere.

Remarks and literature

Literature:

- the original derivation of Regge–Wheeler and Zerilli utilised a full mode decomposition of the <u>linearised</u> Einstein equations
- it took the later work of Moncrief to realise the gauge-invariance of $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$
- the covariant, non-mode decomposed version of these equations is ultimately due to Chaverra, Ortiz and Sarbach

Vanishing of $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$:

- one can show that if $\overset{(1)}{\Phi} = \overset{(1)}{\Psi} = 0$ and $\mathscr{S}$ is sufficiently regular then $\mathscr{S} = \mathscr{K} + \mathscr{G}$

Other appearances:

- Regge–Wheeler and Zerilli – Hung, Keller and Wang
- the Regge–Wheeler equation $\underline{-}$ Dafermos, Holzegel and Rodnianski

The Fackerell–Ipser equation and the gauge-dependent hierarchy

We consider now a hierarchy of <u>scalar</u> quantities $\left(\overset{(1)}{\Phi}, \overset{(1)}{\Psi}, \underset{\vee}{\overset{(1)}{q}}, \overset{(1)}{p}, \overset{(1)}{q}, \underset{\vee}{\overset{(1)}{p}}\right)$ which vanish for all linearised Kerr solutions and are forced by the equations of linearised gravity:

1. to satisfy a hierarchical system of <u>Regge–Wheeler</u> type equations:

$$\psi \in \left(\overset{(1)}{\Phi}, \overset{(1)}{\Psi}, \underset{\vee}{\overset{(1)}{q}}, \overset{(1)}{p}, \overset{(1)}{q}, \underset{\vee}{\overset{(1)}{p}}\right) \implies \Box(r^{-1}\psi) = \mathcal{Z}\psi + F$$

   with F determined by previous members of the hierarchy and $\overset{(1)}{f}$

2. to uniquely determine $\mathcal{T}\overset{(1)}{g}$ where $\mathcal{T}$ is an elliptic angular operator on the 2-spheres foliating $\mathcal{M}$

The linearised metric $\overset{(1)}{g}$ is thus completely determined by the hierarchy $\left(\overset{(1)}{\Phi}, \overset{(1)}{\Psi}, \underset{\vee}{\overset{(1)}{q}}, \overset{(1)}{p}, \overset{(1)}{q}, \underset{\vee}{\overset{(1)}{p}}\right)$ along with the kernel of $\mathcal{T}$.

The key point is that one can derive good estimates for the operators in 1. and 2.!

Remarks

That extract such a hierarchy one exploits a remarkable correspondence between the equations of linearised gravity and *Maxwell's equations in a generalised Lorentz gauge*:

$$\Box A = -j - \mathrm{d}L,$$
$$\mathrm{div} A = -L,$$
$$\mathrm{div} j = 0$$

wtih A,j 1-forms and L a function on $\mathcal{M}$.

In this correspondence the source j and gauge term L are determined by $\overset{(1)}{\Phi}, \overset{(1)}{\Psi}$ and $\overset{(1)}{f}$.

Exploiting the structure of the Maxwell equations thus generates the full hierarchy.

Note one needs the generalised wave gauge condition to extract this correspondence!

III. Gauge-normalisation

First we will require a gauge-normalisation to formulate a statement of quantitative boundedness.

Proposition

Let $\mathscr{S}$ be the smooth solution to the system of gravitational perturbations arising from the smooth seed data set $\mathscr{D}$. Then there exists a pure gauge solution $\mathscr{G}'$ for which the resulting solution

$$\mathscr{S}' := \mathscr{S} + \mathscr{G}' - \mathscr{K}\,,$$

with $\mathscr{K}$ determined explicitly from $\mathscr{D}$, satisfies

i) $\overset{(1)}{\underset{\vee}{q}}{}' = \overset{(1)}{f}{}' = \underline{0}$

ii) *the elliptic operator $\mathcal{T}$ is* <u>*coercive*</u> *on $\mathscr{S}'$.*

Whether a solution is in such a gauge can in fact be detected explicitly from $\mathscr{D}$. The solution $\mathscr{S}'$ is thus said to be *initial-data-normalised*.

To formulate quantitative decay we normalise as follows.

Proposition

Let $\mathscr{S}'$ be initial-data-normalised. Then there exists a pure gauge solution $\mathring{\mathscr{G}}$ for which the resulting solution

$$\mathring{\mathscr{S}}' := \mathscr{S}' + \mathring{\mathscr{G}}$$

satisfies

i) $\overset{(1)}{\underset{\vee}{\mathring{q}}}{}', \overset{(1)}{\mathring{p}}{}', \overset{(1)}{\mathring{q}}{}', \overset{(1)}{\underset{\vee}{\mathring{p}}}{}', \overset{(1)}{\mathring{f}}{}'$ *are given* *<u>explicitly</u>* *by derivatives of* $\overset{(1)}{\Phi}$ *and* $\overset{(1)}{\Psi}$

ii) *the elliptic operators* $\mathcal{T}$ *are coercive on* $\mathring{\mathscr{S}}'$.

This gauge can not be detected from the seed $\mathscr{D}$ alone. The solution $\mathring{\mathscr{S}}'$ is thus said to be a *global-renormalisation* of the solution $\mathscr{S}'$.

Remarks

Motivation behind the gauge:

- adapts and modifies the classical Regge–Wheeler gauge used in the study of the linearised Einstein equations about Schwarzschild
- the presence of the linearised forcing gauge term allows one to adapt this gauge to the equations of linearised gravity
- the modification ensures the solution $\mathring{\mathscr{S}}'$ remains asymptotically flat in evolution and that the 'location of the horizon' is fixed

Utility of the gauge:

- establishing a decay statement for the solution $\mathring{\mathscr{S}}'$ will be relatively simple

The global nature of the gauge:

- necessitates a boundedness statement for $\mathring{\mathscr{G}}$ (equivalently, a boundedness statement for $\mathscr{S}'$) – this is the bulk of our work
- the key insight however is to realise this gauge within the framework of a well-posed formulation of linearised gravity

IV. The linear stability of the Schwarzschild solution

First we have the quantitative boundedness statement.

Theorem 1 (J.)

Let $\mathscr{S}'$ be the smooth initial-data-normalised solution to the system of gravitational perturbations arising from the smooth, asymptotically flat seed data set $\mathscr{D}$. Then on $D^+(\Sigma)$ one has the uniform r-weighted pointwise bounds

$$|r\mathscr{S}'|_{\mathcal{M}} \lesssim \mathbb{D}[\mathscr{D}]$$

with the initial norm finite.

Remarks

On the boundedness statement:

- the boundedness statement is more correctly stated in terms of certain (weighted) energy norms
- this latter statement actually loses derivatives as a consequence of the trapping effect on Schwarzschild

On corollaries to the theorem:

- boundedness for solutions to equations of Regge–Wheeler type and Maxwell's equations on the Schwarzschild exterior

On pointwise decay:

- we in fact obtain a weak rate of dispersion for the solution $\mathscr{S}'$ - this is in 'contrast' to the work of Dafermos, Holzegel and Rodnianksi

Decay of the solution $\mathring{\mathscr{S}}'$

We now have the quantitative statement of linear stability for the Schwarzschild exterior solution.

Let $\tau^\star$ be a function on $\mathcal{M}$ the level sets of which intersect both $\mathcal{H}^+$ and $\mathcal{I}^+$.

Theorem 2 (J.)

Let $\mathscr{S}'$ be as in Theorem 1 and let $\mathring{\mathscr{S}}'$ be its global-renormalisation. Then the pure gauge solution $\mathring{\mathscr{G}}$ satisfies the conclusions of Theorem 1. Moreover, for $\mathring{\mathscr{S}}'$ on $D^+(\Sigma)$ one has the uniform r-weighted decay bounds

$$|r\mathring{\mathscr{S}}'|_{\mathcal{M}} \lesssim \mathbb{D}[\mathscr{D}] \cdot \frac{1}{\sqrt{\tau^\star}}$$

with the initial norm finite. In particular, the solution

$$\mathring{\mathscr{S}} := \mathring{\mathscr{S}}' + \mathscr{K}$$

decays inverse polynomially to the linearised Kerr solution $\underline{\mathscr{K}}$.

Remarks

On the decay statement:

- the decay statement is more correctly stated in terms of certain (weighted) integrated decay (and energy) norms
- the former loses derivatives as a consequence of trapping whereas the latter do not

On corollaries to the theorem:

- decay for solutions to equations of Regge–Wheeler type and Maxwell's equations on the Schwarzschild exterior

On pointwise decay:

- we in fact obtain a stronger rate of dispersion for the solution $\mathring{\mathscr{S}}'$ in the regularity class under consideration

Aside: The scalar wave equation on the Schwarzschild exterior spacetime

Theorem (Dafermos–Rodnianski)

Let ψ be a sufficiently regular solution to $\Box_{g_M}\psi = 0$. Then on $D^+(\Sigma)$ one has the uniform pointwise bounds

i) $|r\psi| \lesssim \frac{1}{\sqrt{\tau^\star}}$.

Fix $\beta_0 > 0$ such that $1 - \beta_0 << 1$ and denote by $\Lambda(\mathcal{M})$ the space of smooth functions on $\mathcal{M}$ supported on the spherical harmonics $l \geq 2$.

Proposition (Angelopoulos–Aretakis–Gajic)

Suppose now $\psi \in \Lambda(\mathcal{M})$ is sufficiently regular. Then on $D^+(\Sigma)$ one has the improved decay

ii) $|r\psi| \lesssim \frac{1}{(\tau^\star)^{2+\frac{\beta_0}{2}}}$.

What are the key ingredients?

The key estimates

i) for any $\tau_2^\star \geq \tau_1^\star$ and $\underline{0 \leq p \leq 2}$ the weighted energy estimates

$$\int_{\Sigma_{\tau_2^\star}} r^p |n\psi|^2 + |\nabla\psi|^2 \lesssim \int_{\Sigma_{\tau_1^\star}} r^p |n\psi|^2 + |\nabla\psi|^2$$

and the weighted integrated decay estimates

$$\int_{\tau_1^\star}^{\tau_2^\star} \int_{\Sigma_{\tau^\star} \cap \{r \leq R\}} |n\psi|^2 + |\nabla\psi|^2 + |\psi|^2 \lesssim \sum_{i=0}^{1} \int_{\Sigma_{\tau_1^\star}} |nT^i\psi|^2 + |\nabla T^i \psi|^2,$$

$$\int_{\tau_1^\star}^{\tau_2^\star} \int_{\Sigma_{\tau^\star} \cap \{r \geq R\}} r^{p-1} |n\psi|^2 + (2-p) r^{p-3} |\nabla\psi|^2 \lesssim \int_{\Sigma_{\tau_1^\star}} r^p |n\psi|^2 + |\nabla\psi|^2$$

with n, ∇ derivatives normal and tangential to the hypersurface $\Sigma_{\tau^\star}$

ii) for $\psi \underline{\in} \Lambda(\mathcal{M})$ and L generating an outgoing null cone $(\underline{r^2}\nabla_L)^2(r\psi)$ satisfies estimates i) with $\underline{p \leq 1 + \beta_0}$

Weaker r-weighted versions of the estimates in i) are analogues of what we prove for the solution $\mathscr{S}'$. Conversely, for the solution $\mathring{\mathscr{S}}'$ we prove an analogous full hierarchy of estimates found in i)+ii).

Remarks

On estimates i):

- to prove the $p = 0$ version of the energy estimate requires exploiting the celebrated red-shift effect on Schwarzschild in conjuction with the (degenerate!) conserved energy associated to the Killing field T
- (some) loss of derivatives in the integrated local energy estimate is unavoidable and arises from the existence of trapped null geodesics at $r = 3M$ on Schwarzschild which is an obstruction to decay in the geometric optics approximation (Sbierski)

On estimates ii):

- the second order radiation field is intimately related to our notion pointwise aymptotic flatness for solutions to the equations of linearised gravity

V. Outline of the proofs: Boundedness and decay for solutions to equations of Regge–Wheeler type on the Schwarzschild exterior spacetime

Theorem 3 (J.)

Let $\Psi \in \Lambda(\mathcal{M})$ be a solution to the Regge–Wheeler type equation

$$\Box\left(r^{-1}\Psi\right) = \mathcal{Z}\Psi + F$$

with $F \in \Lambda(\mathcal{M})$. Then assuming sufficient regularity the following hierarchy of estimates hold:

$$F = 0 \implies |\Psi| \lesssim \frac{1}{(\tau^\star)^{2+\frac{\beta_0}{2}}},$$

$$|F| \lesssim \frac{1}{(\tau^\star)^{2+\frac{\beta_0}{2}}} \implies |\Psi| \lesssim \frac{1}{(\tau^\star)^{\frac{1}{2}+\frac{\beta_0}{2}}},$$

$$|F| \lesssim \frac{1}{(\tau^\star)^{\frac{1}{2}+\frac{\beta_0}{2}}} \implies |\Psi| \lesssim \frac{r^{\frac{1}{2}-\frac{\beta_0}{4}}}{(\tau^\star)^{\frac{\beta_0}{4}}}.$$

Applying this Theorem to the hierarchy associated to the solutions $\mathscr{S}'$ and $\mathring{\mathscr{S}}$ ultimately yields Theorems 1 and 2.

Remarks

On the statement:

- the boundedness and decay statements are more correctly stated in terms of weighted energy and integrated decay norms

On previous work:

- for the Regge–Wheeler equation the (weaker) decay statement was originally obtained by Holzegel with earlier work of Blue–Soffer
- for the Zerilli equation the (weaker) decay statement was obtained by independently by J. and Hung–Keller–Wang

On extra difficulties when implemented in Theorem 1:

- to get the correct regularity for the inhomogeneous terms one must perform a (complicated!) renormalisation procedure

On the derivative loss of Theorem 1:

- the loss due to trapping gets amplified as one ascends the hierarchy

A restricted nonlinear stability conjecture

Conjecture (Dafermos–Holzegel–Rodnianski, 2016)

Let (Σ_M, h_M, k_M) *be the induced data on a spacelike asymptotically flat slice of the Schwarzschild solution of mass* M *crossing the future horizon and bounded by a trapped surface. Then in the space of all nearby vacuum data* (Σ, h, k)*, in a suitable norm, there exists a codimension-3 subfamily for which the corresponding maximal vacuum Cauchy development* (M, g)*:*

i) *contains a black-hole exterior region characterized as the past* $J^-(\mathcal{I}^+)$ *of a complete future null infinity* $\mathcal{I}^+$

ii) *is bounded by a non-empty future affine-complete event horizon* $\mathcal{H}^+$

iii) *in* $J^-(\mathcal{I}^+)$ *the metric remains close to* g_M *and moreover asymptotically settles down to a nearby Schwarzschild metric* $g_{M'}$ *at suitable inverse polynomial rate.*