Seminar on Mathematical General Relativity
LJLL, Sorbonne U./UPMC/Paris VI/Jussieu

General Relativity at the Subatomic Scale

Dirac Equation on Mildly Singular Spacetimes

A. Shadi Tahvildar-Zadeh

Department of Mathematics
Rutgers –New Brunswick

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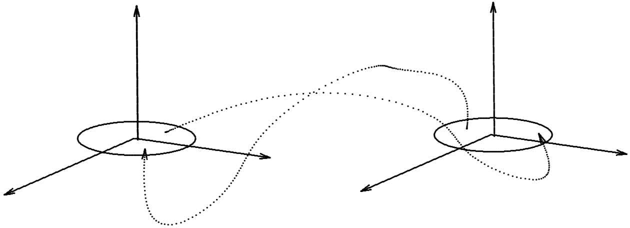
Some Recent Results in GRQM

1. A. S. Tahvildar-Zadeh, “On the static spacetime of a single point charge", *Reviews in Mathematical Physics* **23**(3): 309–346 (2011).
2. A. S. Tahvildar-Zadeh, “On a zero-gravity limit of Kerr–Newman spacetimes and their electromagnetic fields," 22 pp. (2012) appeared in *Jour. Math. Phys.* (2015) [`arXiv:1410.0416`].
3. M. K.-H. Kiessling & A. S. Tahvildar-Zadeh, “The Dirac point electron in zero-gravity Kerr–Newman spacetime," 47 pp., appeared in *Jour. Math. Phys.* (2015) [`arXiv:1410.0419`].
4. M. K.-H. Kiessling & A. S. Tahvildar-Zadeh, “A novel quantum-mechanical interpretation of the Dirac equation,” 47 pp., appeared in *Jour. of Phys. A* (2016) [`arXiv:1411.2296`].
5. M. K. Balasubramanian, “Scalar fields and Spin-1/2 Fields on Mildly Singular Spacetimes," 71 pp., Ph.D. Dissertation, Rutgers –New Brunswick (May 2015).

Sommerfeld's Space(-time)

- Sommerfeld (1896) Generalization of Riemann surfaces to 3-D
- Two copies of $\mathbb{R}^3$ cross-glued at a disk (of radius a):



- There is a coordinate system that covers this manifold in a single chart: Oblate Spheroidal Coordinates (r, θ, φ)
- Add a time dimension to get a static spacetime.
- It is flat (away from the ring) $\implies$ it is a solution of Einstein Vacuum Equations
- It's a limiting member of **two** known families of solutions to E.V.E.
- Points on the ring are conical singularities for the metric

Appell's Magical Electromagnetic Field

- The Coulomb potential: $\phi(\mathbf{x}) = \frac{Q}{|\mathbf{x}-\mathbf{x}_0|} = \frac{Q}{\sqrt{(\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}}$
- Harmonic function on $\mathbb{R}^3 \setminus \{\mathbf{x}_0\}$.
- Static electric field of a point charge located at $\mathbf{x}_0$: $\mathbf{E} = -\nabla_{\mathbf{x}}\phi$.
- Paul Appell (1888) replaced $\mathbf{x}_0$ by $\mathbf{z}_0 = \mathbf{x}_0 + i\mathbf{y}_0 \in \mathbb{C}^3$!
- $\Phi(\mathbf{x}) = \frac{Q}{\sqrt{(\mathbf{x}-\mathbf{z}_0)\cdot(\mathbf{x}-\mathbf{z}_0)}}$
- Set $\mathbf{E} + i\mathbf{B} = -\nabla_{\mathbf{x}}\Phi$.
- New, multi-valued solution of Maxwell's equations!
- Singular on a ring of radius $|\mathbf{y}_0|$ centered at $\mathbf{x}_0$, with $\mathbf{n} = \mathbf{y}_0/|\mathbf{y}_0|$ the normal to the plane of the ring.
- The solution becomes single-valued when its domain is extended to the two-sheeted Sommerfeld space.
- The singular ring appears to be positively charged in one sheet, and negatively charged in the other sheet!!

Sommerfeld, Evans and Zipoy

- Sommerfeld's generalization of Riemann surfaces to three dimensions.
- The space on which a multi-valued harmonic function become single-valued.
- Example: The Appell potential.
- Connection with the diffraction problem and Kelvin's method of images.
- G. Evans (1940's) generalized Sommerfeld's work in every possible direction!
- Evans proved existence of Green's function for branched Riemann spaces
- Zipoy in 1964 found the first multi-sheeted solution family of Einstein's vacuum equations.
- Sommerfeld space is a limiting member of the Zipoy family.

Particle-like solutions of Einstein's equations

- The famous “Bridge" paper of Einstein & Rosen
- They glued two copies of exterior Schwarzschild together at their horizons, to obtain a spacetime free of singularities. But why?
- Title: “The Particle Problem in the General Theory of Relativity", *Phys. Rev.* **48** (1935).
- “To what extent can general relativity account for atomic structure of matter and for quantum effects?"
- What they wanted: solutions of Einstein's equations that have particle-like features
- Problem: They abhor singularities, so points are not allowed!
- *“Writers have occasionally noted the possibility that material particles might be considered as singularities of the field. This point of view, however, we cannot accept at all...Every field theory must adhere to the fundamental principle that singularities of the field are to be excluded."*

Particles as Bridges

- Einstein and Rosen's important insight:
- *"These solutions involve the mathematical representation of physical space by* **a space of two identical sheets, a [charged] particle being represented by a 'bridge' connecting these sheets.***"*
- Two crazy ideas of E.&R.:
  1. Particles can be represented within GR as bridges connecting two vacuums
  2. Space can be multi-sheeted (e.g. like a Riemann surface)
- NOTE: The E-R bridge cannot be crossed!
- Our observation: **The ring singularity of the Sommerfeld Space with the Appell field on it represents a two-faced particle that connects the two sheets of the spacetime.** So the E-R dream comes true!

Particles as Singularities

- Herman Weyl's Singularity Theory of Matter (1921):
- *"Matter now appears as a real singularity of the field (...) In the general theory of relativity the world can possess arbitrary (...) connectedness: nothing excludes the assumption that in its Analysis-Situs [i.e. topological] properties it behaves like a four dimensional Euclidean continuum, from which different tubes of infinite length in one dimension are cut off."*
- Weyl's crazy ideas:
  1. A particle *IS* a singularity of space
  2. These singularities need to be excised from space: Matter *is* where space *isn't.*
- Note: This also leads to spacetimes with non-trivial topology.
- Key Question: Can a quantum law of motion be formulated for the ring-like particle?

The Singularity Problem in QM

- Quantum law of motion for charged, point-like particles interacting with an electromagnetic field?
- Classical obstruction: Maxwell-Lorentz Electrodynamics:
  1. Charges and currents are sources for the EM field (Maxwell)
  2. EM Field act on charges and make them move (Lorentz)
- A classical problem in electrodynamics of point particles:
  1. EM field of a point charge is undefined at the location of the charge
  2. Self-Energy of a point charge is infinite
- Consequences:
  1. M-L electrodynamics is not well-defined for point charges
  2. Electrovacuum spacetimes are highly singular, e.g. Reissner-Nordstrom (1916), Kerr-Newman family (1965).
- Note: These problems are NOT resolved (in a mathematically rigorous way) by currently known extensions of QM (QED, QFT, ...)
- **Interesting Fact:** Sommerfeld-Appell is also a limiting member of the Kerr-Newman family, arising in the limit $G \to 0$.

Special-relativistic Quantum Mechanics

- What do you see when you heat up a gaseous element until it glows, and look at it through a prism?
- Spectral lines.
- Why not a rainbow?
- Bohr and the quantum revolution
- Orbitals are Energy eigenstates!
- Schrödinger's equation reproduces Bohr's spectrum!
- Problem: Schrödinger's equation is non-relativistic
- Sommerfeld's (1924) relativistic corrections to Bohr's spectrum
- Dirac 1928: New equation for the wave function of the electron
- Dirac's equation is Lorentz-invariant
- Dirac's spectral lines came out to be exactly the same as Sommerfeld's corrected version of Bohr's.
- Schrödinger and Dirac shared a Nobel Prize!

Special-Relativistic Hydrogen

- Dirac's Equation for Electron in Proton's electrostatic Field, in Hamiltonian form: $i\partial_t\Psi = \mathbf{H}\Psi$
- Eigen-functions: $\Psi(t,\mathbf{x}) = e^{-iEt}\psi(\mathbf{x}) \implies \mathbf{H}\psi = E\psi$
- $$\sigma_{disc}(H_{Dirac}) = \frac{m}{\sqrt{1+\left(\frac{e^2}{n-|k|+\sqrt{k^2-e^4}}\right)^2}}$$
- $n = 1,2,3,\ldots, \qquad k = -n,-n+1,\ldots,-1,1,\ldots,n-1$
- Spectroscopy. $s,p,d,f,g,\ldots$ Orbitals. Degenerate and non-degenerate states. Hyperfine splitting. Lamb shift. QED. More Nobel prizes!
- $\sigma_{cont}(H_{Dirac}) = (-\infty,-m] \cup [m,\infty)$
- Negative Continuum: Dirac's Sea, Hole Theory, Positron

General-relativistic Hydrogen?

- Einstein's dream
- Gravity is much weaker than electromagnetism, so adding gravitational effects to Dirac's equation should shift spectral lines by a tiny amount, if at all
- Nasty Surprise: general-relativistic Dirac Hamiltonian has NO discrete spectrum. (Belgiorno et al, Finster et al.)
- Pathologies of well-known solutions to Einstein's equations
- Likely culprit: Infinite self-energy of point charges (linear electromagnetics) causes the spacetime to be infinitely curved close to the charge.
- A possible remedy: There are **other electromagnetic vacuum laws** that can give rise to **finite self-energies**, and thus milder singularities for the spacetime
- Another option: Take the **zero-gravity limit** and work with the **topological remnants** of gravity.

Nonlinear Electromagnetics

- General Maxwell's equations involve four fields: $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$.
- Need constitutive relations to close: $\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B}), \mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B})$.
- Maxwell's vacuum law: $\mathbf{E} = \mathbf{D}, \mathbf{H} = \mathbf{B}$. Standard electromagnetics
- Coulomb potential and the electrostatic energy of a point charge
- Born's idea: $\mathbf{E} = \mathbf{D}/\sqrt{1 + |\mathbf{D}|^2}$
- Generalized to Born-Infeld's nonlinear electromagnetics, giving rise to **finite self-energy of point charges.**
- Einstein-Maxwell-Born-Infeld equations:
  The Hoffmann spacetime. $m_{ADM} =$Electrostatic energy. **Has a Conical singularity at the center.**
- Note: This is the **mildest** singularity possible (T.-Z. 2011)
- Dirac equation on Hoffmann spacetime:
  Self-adjointness of the Hamiltonian, and the **existence of discrete spectrum** (Moulik Balasubramanian, 2015)
- The Great Challenge: going beyond spherical symmetry for Einstein-Maxwell-Born-Infeld.

Linear, Higher-Order Electromagnetics

- Bopp-Lande-Thomas-Podolsky (BLTP) theory
- $\mathbf{H} = (1 + \varkappa^{-2}\Box)\mathbf{B}$
- $\mathbf{D} = (1 + \varkappa^{-2}\Box)\mathbf{E}$
- $\varkappa$ = Bopp parameter, induces Klein-Gordon smoothing effects
- Self-energy of a point charge is finite
- EM field of a point charge is still singular, but only mildly so (similar to Born-Infeld)
- Maxwell's equations with BLTP vacuum law is a *linear* theory!
- **THEOREM** (Kiessling and T.-Z., 2017) Local well-posedness of the *joint* initial value problem for the Maxwell-BLPT system with N point charges. (Self-force is NOT the Lorentz force)
- Coupling to gravity is now possible, and may be amenable to the ***G*-perturbation method** (Erik de Amorim, work in progress)

General-relativistic Zero-Gravity Quantum Mechanics

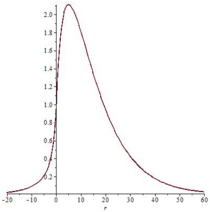
- Two sources of G-dependence for Einstein geometries
- The coupling constant G
- The dimension conversion constant G
- Example: Reissner-Nordström, in spherical coordinates $g_{RN} = \mathrm{diag}(-f(r), \frac{1}{f(r)}, r^2, r^2 \sin^2 \theta)$
- $f(r) = 1 - \frac{2GM}{c^2 r} + \frac{GQ^2}{c^4 r^2}$
- Take the $G \to 0$ limit, recover Minkowski space
- There are other famous solutions of Einstein's Eqs, where the zero-G limit does NOT give you back Minkowski space
- Nontrivial geometry goes away, nontrivial topology remains
- The Kerr-Newman solution (1965), and its Maximal Analytical Extension (1968)
- The Magic Hoop Returns: Appell-Sommerfeld is the zero-gravity limit of the maximal-analytically-extended Kerr-Newman solution!

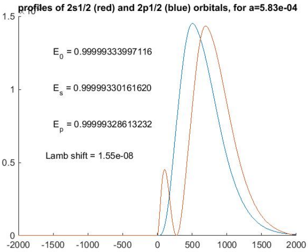
The Zero-G Limit comes to Dirac's Rescue

- Dirac's equation for an electron in the zero-G Kerr–Newman (zGKN) spacetime
- The zero-G limit of a positively charged Reissner- Nordström spacetime yields Minkowski spacetime with a positive Coulomb singularity — Nothing New!: *Back to Special-Relativistic Hydrogen*
- The zero-G limit of the maximally extended Kerr-Newman spacetime yields a flat, but topologically non-trivial, multi-sheeted electromagnetic spacetime.
  This does yield Something New! .... and Interesting:
  Zero-G Kerr-Newman Hydrogen exists!

Some of our results on zero-G Hydrogen spectrum

- The Dirac Hamiltonian on zero-G Kerr Newman spacetime is essentially self-adjoint
- The spectrum is symmetric about zero.
- Discrete spectrum is non-empty under some smallness conditions.
- The continuous spectrum is $(-\infty, -m] \cup [m, \infty)$
- Profile of the positive energy ground state shows the tiger's tail!





- Excited states. Numerical approximation. Hyperfine splitting and Lamb shift without QED!

Back to Einstein, Rosen, and Weyl

- Recall: The ring singularity of the zGKN spacetime
- It connects the two sheets of the (otherwise vacuum) spacetime
- It's the locus of singularities for the metric, so the two-sheeted spacetime is defined outside a timelike 2-dim tube (circle x real line)
- It's positively charged in one sheet, negatively charged in the other
- Our radically new idea: Electron and Positron are not distinct particles but in fact "two different sides of the same coin"
- This resolves the paradox that Dirac's equation "for the electron" also seems to describe "a positron" in many situations, while it is a true one-particle equation.

Topo-spin

- We introduce the notion of **TOPOLOGICAL SPIN** in analogy to Heisenberg's iso-spin.
- we identify the ring singularity of the zGKN spacetime with a **two-faced particle**, one that appears as an electron in one sheet and a positron in another sheet
- The radius of the ring = anomalous magnetic moment of the electron
- Can we formulate a quantum law of motion for the (center of the) ring?
- YES! By relativity, it is the one that we have discussed! (at least for quasi-static motions)
- Anti-symmetry of the Dirac Hamiltonian with respect to topo-spin flips gives rise to the *matter-antimatter duality*

Proof of symmetry of spectrum

- Let H be a matrix with a real eigenvalue E and a corresponding eigenvector Ψ
$$H\Psi = E\Psi$$
- Suppose there is another matrix C that anti-commutes with H, and $C\Psi \neq 0$:
$$CH + HC = 0$$
- Then $-E$ is also an eigenvalue of H, with eigenvector $C\Psi$.
- We found a C that does the job for our Hamiltonian H.
- C is a topo-spin flip! It uses the double-sheetedness of spacetime.
- Eigenfunctions with positive energy are 99% supported in one sheet, and those with negative energy are 99% supported in the other sheet

Summary

- Nonlinear Electrodynamics is hard, but definitely worth doing!
- Topologically non-trivial spacetimes should be taken seriously
- The Dirac equation on zGKN is well-posed; Naked singularity means no harm!
- The Dirac Hamiltonian on zGKN has symmetric spectrum with scattering and bound states
- Novel proposal: Dirac's equation describes a single "particle / anti-particle" structure: two "topo-spin" states

Our To-Do List

- Characterize the discrete Dirac spectrum on zGKN completely!
- Study zero-G general-relativistic Dirac spectrum for "Positronium" (Bound states of an electron and a positron): 4-sheeted space with two ring singularities + a *multi*-Appell field!
- Limit of infinitely many rings may exhibit "ferro-topological phase transition": (Explanation of broken particle / anti-particle symmetry in the universe?).
- Turn gravity back on! (EMBI or EMBLTP, G-Perturbation method?)
- Bring on the photons: mass-less spin one particles with no longitudinal wave degree of freedom. (Singularities of the electromagnetic field?) What is the wave function and the wave equation?
- Photon-electron interactions: Compton scattering, Emission / Absorption, QED without divergences?
- Model other particles (quarks, etc) as singularities of Riemann spaces branched over knots.

Proof of existence of discrete spectrum

- $H\Psi = E\Psi$. Find $E \in (-m, m)$ and $\Psi : \text{Somm}_1 \to \mathbb{C}^4$
- Chandrasekhar (1976) Separation of variables in Oblate Spheroidal Coordinates (r, θ, φ).
- Prüfer-ed Chandrasekhar Ansatz for $\Psi \in \mathbb{C}^4$.
- $$\Psi(r,\theta,\varphi) = R(r)S(\theta)e^{i\kappa\varphi}\begin{pmatrix} \cos(\Theta(\theta)/2)e^{+i\Omega(r)/2} \\ \sin(\Theta(\theta)/2)e^{-i\Omega(r)/2} \\ \cos(\Theta(\theta)/2)e^{-i\Omega(r)/2} \\ \sin(\Theta(\theta)/2)e^{+i\Omega(r)/2} \end{pmatrix}$$
- $$\begin{cases} d\Omega/dr = 2\frac{mr}{\Delta}\cos\Omega + 2\frac{\lambda}{\Delta}\sin\Omega + 2\frac{a\kappa+\gamma r}{\Delta^2} - 2E \\ d(\ln R)/dr = \frac{mr}{\Delta}\sin\Omega - \frac{\lambda}{\Delta}\cos\Omega \end{cases}$$
  $$\begin{cases} d\Theta/d\theta = 2(\lambda - ma\cos\theta\cos\Theta + \left(aE\sin\theta - \frac{\kappa}{\sin\theta}\right)\sin\Theta) \\ d(\ln S)/d\theta = -ma\cos\theta\sin\Theta - \left(aE\sin\theta - \frac{\kappa}{\sin\theta}\right)\cos\Theta. \end{cases}$$
- $\Psi \in L^2$ iff:
  $$\begin{cases} \Omega(-\infty) = -\pi + \cos^{-1}(E), \quad \Omega(\infty) = -\cos^{-1}(E) \\ \Theta(0) = 0, \qquad \Theta(\pi) = -\pi. \end{cases}$$

Flows on a finite cylinder

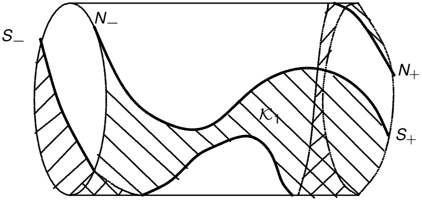
- $$\begin{cases} \dot{\theta} &= \sin\theta \\ \dot{\Theta} &= -2a\sin\theta\cos\theta\cos\Theta + 2aE\sin^2\theta\sin\Theta - 2\kappa\sin\Theta \\ & \quad + 2\lambda\sin\theta \end{cases}$$
  $$\begin{cases} \dot{\xi} &= \cos^2\xi \\ \dot{\Omega} &= 2a\sin\xi\cos\Omega + 2\lambda\cos\xi\sin\Omega + 2\gamma\sin\xi\cos\xi \\ & \quad + 2\kappa\cos^2\xi - 2aE \end{cases}$$
- Parameter-dependent flow on a cylinder

$$\begin{cases} \dot{x} &= f(x) \\ \dot{y} &= g_\mu(x,y) \end{cases} \qquad (x,y) \in [x_-, x_+] \times \mathbb{S}^1$$

- Two equilibrium points on each boundary: $f(x_-) = f(x_+) = 0$ and $g_\mu(x_\pm, y) = 0 \implies y \in \{s_\pm, n_\pm\}$

Looking for Heteroclinic Orbits

- Nodes $N_\pm = (x_\pm, n_\pm)$ and Saddles $S_\pm = (x_\pm, s_\pm)$
- Existence of N_- to S_+ and S_- to N_+ connecting orbits (stable/unstable and center manifold theory.)
- Existence of L^2 eigenfunction iff there is an orbit connecting the two saddles: S_- to S_+.
- The Corridor formed by the two SN connectors.



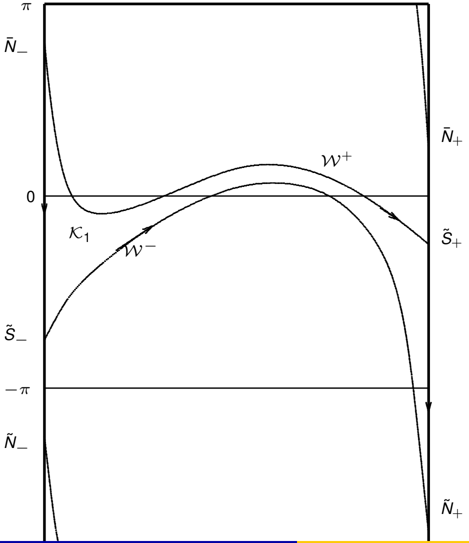
- Saddle-Saddle connector exists iff the corridor collapses.

Topological Methods in Dynamical Systems

- μ-dependent flow on a cylinder
- **a**(μ) = area of corridor
- **w**(μ) = winding number of corridor
- **a** is a continuous function of μ
- **a** > 0 iff **w** ≥ 1 (Green's theorem).
- **a** < 0 iff **w** ≤ 0.
- **a** $= 0$ iff corridor is empty (i.e. there is a saddle connection.)
- Construction of barriers to prove existence of corridors with given winding number.

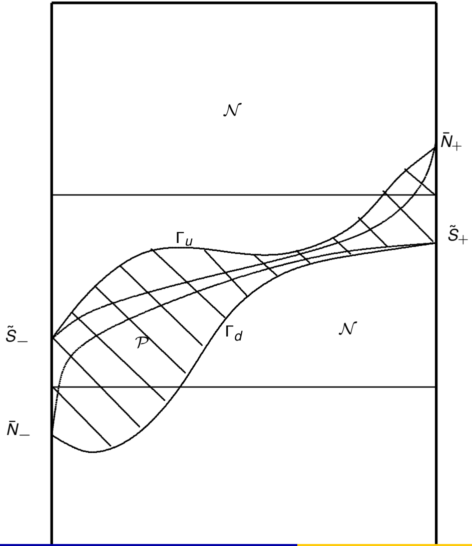
Area and Winding Number for Corridors

Working in the universal cover of the cylinder:

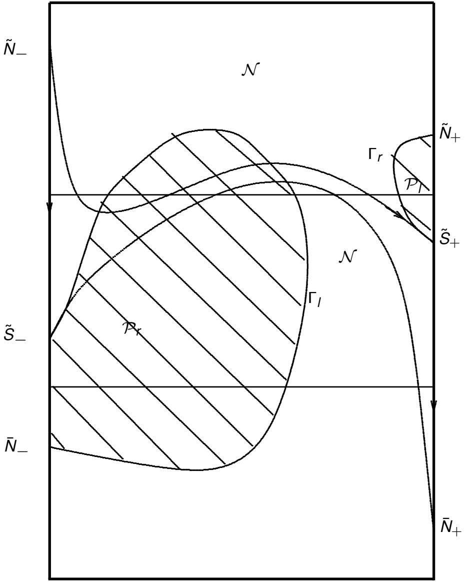


Topology of Nullclines

Orbits must increase while in the shaded region



Change in Nullcline Topology and Corridor Winding



Barrier construction