# On the linear stability of Kerr black holes

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# Einstein vacuum equation

We are interested in the global behavior of solutions of

$$\mathrm{Ric}(g) = 0,$$

where $g$ is a Lorentzian metric $(+---)$ on a 4-manifold $M$.

Here: study perturbations of special solutions.

# Special solutions of $\mathrm{Ric}(g) = 0$

1. Minkowski space.

$$M = \mathbb{R}_t \times \mathbb{R}^3_x,$$
$$g_{(0,0)} = dt^2 - dx^2 = dt^2 - dr^2 - r^2 g_{\mathbb{S}^2}.$$

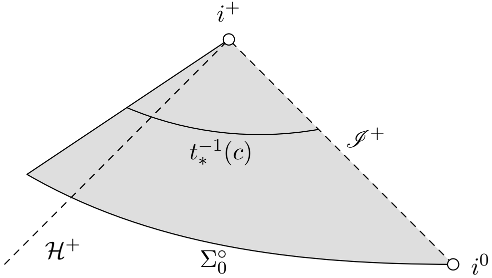
2. Schwarzschild black holes (mass $\mathfrak{m} > 0$).

$$M = \mathbb{R}_t \times (0,\infty)_r \times \mathbb{S}^2,$$
$$g_{(\mathfrak{m},0)} = \left(1 - \frac{2\mathfrak{m}}{r}\right) dt^2 - \left(1 - \frac{2\mathfrak{m}}{r}\right)^{-1} dr^2 - r^2 g_{\mathbb{S}^2}$$
$$= g_{(0,0)} + \mathcal{O}(r^{-1}).$$

# Illustration of the Schwarzschild metric

$$b_0 = (\mathfrak{m}_0, 0), \quad g_{b_0} = \left(1 - \frac{2\mathfrak{m}_0}{r}\right) dt^2 - \left(1 - \frac{2\mathfrak{m}_0}{r}\right)^{-1} dr^2 - r^2 g_{\mathbb{S}^2}.$$

Regge-Wheeler coordinate : $r_* = r + 2\mathfrak{m}_0 \log(r - 2\mathfrak{m}_0)$.
$t_* = t + r_*$ near $\mathcal{H}^+$ $(r = 2\mathfrak{m}_0)$, $t_* = t - r_*$ near $\mathscr{I}^+$.
$M = \mathbb{R}_{t_*} \times X, \quad X = [r_-, \infty) \times \mathbb{S}^2, \ r_- \in (0, 2\mathfrak{m}_0)$.



# Special solutions of $\mathrm{Ric}(g) = 0$, continued

3. Kerr black holes
   (mass $\mathfrak{m} > 0$, angular momentum $\mathfrak{a} \in \mathbb{R}^3$, $a = |\mathfrak{a}|$).

$$
\begin{aligned}
g_{(\mathfrak{m},\mathfrak{a})} &= \frac{\Delta_b - a^2 \sin^2\theta}{\varrho_b^2} dt^2 + \frac{4a\mathfrak{m}r\sin^2\theta}{\varrho_b^2} dt d\varphi \\
&\quad - \frac{(r^2+a^2)^2 - \Delta_b a^2 \sin^2\theta}{\varrho_b^2} \sin^2\theta d\varphi^2 - \varrho_b^2\Big(\frac{dr^2}{\Delta_b} + d\theta^2\Big) \\
&= g_{(\mathfrak{m},0)} + \mathcal{O}(r^{-2}), \\
&\quad \Delta_{(\mathfrak{m},\mathfrak{a})} = r^2 - 2\mathfrak{m}r + a^2, \quad \varrho^2_{(\mathfrak{m},\mathfrak{a})} = r^2 + a^2\cos^2\theta.
\end{aligned}
$$

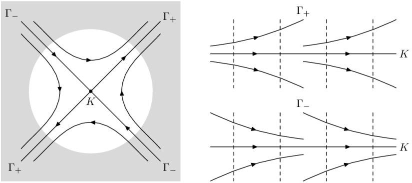
Consider slowly rotating Kerr black holes:

$$b := (\mathfrak{m}, \mathfrak{a}) \approx b_0 = (\mathfrak{m}_0, 0) \text{ on } M = \mathbb{R}_{t_*} \times X.$$

$g_b$ for such $b$ is a smooth family of stationary metrics on $M$.

## Kerr solution continued

- The Kerr metric is asymptotically Minkowskian.
- There exist trapped null geodesics. $r$-normally hyperbolic trapping for each $r$ (stable property with respect to perturbations).




- There doesn't exist any global timelike Killing vector field outside the black hole. Consequence : no conserved positive quantity for the wave equation.

Analogous solution for positive cosmological constant : De Sitter Kerr metric. The De Sitter Kerr metric is asymptotically De Sitter.

Only special solutions or real ?

# Initial value problem for $\mathrm{Ric}(g) = 0$

Given on $\Sigma = t^{-1}(0) \subset M$:

- $\gamma$: Riemannian metric on $\Sigma$,
- $k$: symmetric 2-tensor on $\Sigma$.

Find:

- Lorentzian metric $g$ on $M$, $\mathrm{Ric}(g) = 0$,
- $\tau(g) := (-g|_\Sigma, \mathrm{II}^{\mathrm{g}}_\Sigma) = (\gamma, \mathrm{k})$.

Necessary and sufficient for local existence: constraint equations on $(\gamma, k)$. (Choquet-Bruhat '52.)

Example

For $(\gamma, k) = (\gamma_b, k_b) := \tau(g_b)$, the solution of the initial value problem is $g_b$.

# Kerr black hole stability

Kerr :

Theorem (Klainerman, Szeftel, Giorgi, Shen '22)

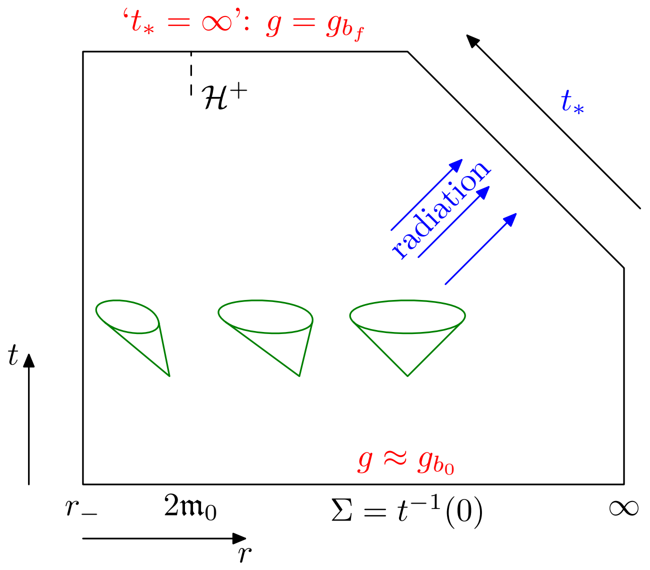
*The future globally hyperbolic development of a general, asymptotically flat, initial data set, sufficiently close (in a suitable topology) to a Kerr* $(\mathfrak{a}_0, \mathfrak{m}_0) = b_0$ *initial data set, for sufficiently small* $\frac{|a_0|}{\mathfrak{m}_0}$*, has a complete future null infinity* $\mathscr{I}^+$ *and converges in its causal past* $J^-(\mathscr{I}^+)$ *to another nearby Kerr spacetime with parameters* $b_f$ *close to the initial ones* $b_0$.

De Sitter Kerr:

Theorem (Hintz, Vasy '16)

*In an equivalent situation for De Sitter Kerr there exists* $g$ *such that*

$$g = g_{b_f} + \tilde{g}, \quad |\tilde{g}| \lesssim e^{-\beta t_*}, \, \beta > 0.$$



'$t_* = \infty$': $g = g_{b_f}$
$\mathcal{H}^+$
$t_*$
radiation
$g \approx g_{b_0}$
$t$
$r_-$
$2\mathfrak{m}_0$
$\Sigma = t^{-1}(0)$
$\infty$
$r$

## $\Lambda > 0$ versus $\Lambda = 0$.

Toy model of linearized Einstein equations around (De Sitter) Schwarzschild : wave equation on scalars :

$$(\partial_t^2 + P)u = 0.$$

- De Sitter Schwarzschild : $P$ similar to a Laplace Beltrami operator on a manifold with two asymptotically hyperbolic ends : meromorphic extension of the resolvent on suitable spaces (Mazzeo-Melrose, Guillarmou).
- Schwarzschild : one asymptotically euclidean end : close to zero, resolvent only $H^k$ down to the real axis (Bony-H., Guillarmou-Hassell, Wunsch-Vasy, Vasy,...)

# Kerr stability: main issues

1. Find final black hole parameters $(\mathfrak{m}_f, \mathfrak{a}_f)$.
2. Diffeomorphism invariance: $\mathrm{Ric}(g) = 0 \Rightarrow \mathrm{Ric}(\Phi^* g) = 0$.
   - gauge fixing;
   - track location of black hole in chosen gauge.
3. Because of the weak decay for the linearized problem, the precise structure of the nonlinearity is needed.

In Hintz–Vasy: use Newton-type iteration scheme; naively $g_0 = g_{b_0}$,

$$\begin{cases} D_{g_0}\mathrm{Ric}(h_0) = -\mathrm{Ric}(g_0), \\ \text{initial data for } h_0 \text{ on } \Sigma, \end{cases} \implies \quad g_1 = g_0 + h_0, \quad \text{etc.}$$

Idea: read off improved guess of final black hole parameters and location/velocity from asymptotic behavior of $h_0$.

# Linear stability (modulo gauge)

Consider black hole parameters $b \approx b_0 = (\mathfrak{m}_0, 0)$.

**Theorem (H.–Hintz–Vasy '19)**

*Let $\gamma', k'$ be symmetric 2-tensors on $\Sigma = t^{-1}(0)$ satisfying the linearized constraint equations and*

$$|\gamma'| \lesssim r^{-1-\alpha}, |k'| \lesssim r^{-2-\alpha}, 0 < \alpha < 1,$$

*(and similar bounds for derivatives). Then there exists a symmetric 2-tensor $h$ on $M$ such that*

$$D_{g_b}\mathrm{Ric}(h) = 0, \quad D_{g_b}\tau(h) = (\gamma', k'),$$

*which decays to a linearized Kerr metric,*

$$h = g_b'(b') + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}, \quad \left( g_b'(b') := \frac{\mathrm{d}}{\mathrm{d}s} g_{b+sb'} \Big|_{s=0} \right).$$

# Gauge fixing

Eliminate diffeomorphism invariance: impose extra condition on $g$:

$$W(g) = \Box_{g,g_b}\mathbf{1} \ (= \text{1-form in } g,\ \partial g) = 0.$$

Then ('DeTurck trick'):

$$\begin{cases} \mathrm{Ric}(g) = 0, \\ W(g) = 0, \\ \text{initial data} \end{cases} \iff \text{IVP for } P(g) := \mathrm{Ric}(g) - \delta_g^* W(g) = 0.$$

Linearized version:

$$\begin{cases} D_{g_b}\mathrm{Ric}(h) = 0, \\ D_{g_b}W(h) = 0, \\ \text{initial data} \end{cases} \iff \text{IVP for } D_{g_b}P(h) \ \left(\approx \frac{1}{2}\Box_{g_b} h\right) = 0.$$

# Main theorem

Let $L_b := D_{g_b}P$. Study $L_b h = 0$ with general initial data.

**Theorem (H.–Hintz–Vasy '19)**

*Let* $\alpha \in (0,1)$, *and let* $h_0, h_1 \in \mathcal{C}^\infty(\Sigma; S^2 T^*_\Sigma M)$,

$$|h_0| \lesssim r^{-1-\alpha}, \quad |h_1| \lesssim r^{-2-\alpha}, \, 0 < \alpha < 1$$

*(and similar bounds for derivatives). Let*

$$\begin{cases} L_b h = 0, \\ (h|_\Sigma, \ \mathcal{L}_{\partial_t} h|_\Sigma) = (h_0, h_1). \end{cases}$$

*Then there exist* $b' \in \mathbb{R} \times \mathbb{R}^3$ *and* $V \in$ *fixed 8-dimensional space of vector fields on* $M$ *such that*

$$h = g'_b(b') + \mathcal{L}_V g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}.$$

# Main theorem, continued

$$h = g'_b(b') + \mathcal{L}_V g_b + \tilde{h}, \quad |\tilde{h}| \lesssim t_*^{-1-\alpha+}.$$

Here,

$$\begin{aligned} V \in \operatorname{span}\big\{&\text{asymptotic translations: } \partial_{x^i} + \mathcal{O}(r^{-1}), \\ &\text{asymptotic boosts: } t\partial_{x^i} - x^i\partial_t + \text{l.o.t.}\big\} \\ &+ \text{two non geometric vector fields.} \end{aligned}$$

Can read off:

- change of black hole parameters,
- movement of black hole in chosen gauge.

Remark: upon given up one order of decay one can choose $V$ in a 6 dimensional space consisting only of the vector fields with clear geometric interpretation.

# Prior work

- Andersson–Bäckdahl–Blue–Ma '19: initial data with fast decay ($\alpha > 5/2$; then $b' = 0$)
- Dafermos–Holzegel–Rodnianski '16: $b = b_0$
- Johnson, Hung–Keller–Wang '16–'18: $b = b_0$
- Finster–Smoller '16 (no decay rate).

# Strategy of proof

Recast as

$$L_b h = f.$$

Work on spectral side:

$$h(t_*, x) = \frac{1}{2\pi} \int_{\operatorname{Im}\sigma = C} e^{-i\sigma t_*} \widehat{L_b}(\sigma)^{-1} \hat{f}(\sigma, x)\, d\sigma.$$

Shift contour to $C = 0$.

Uses/is related to: Melrose, Vasy–Zworski, Vasy; Wunsch–Zworski, Dyatlov, Hintz, Vasy, Guillarmou–Hassell, Bony–H.

# $1^{\text{st}}$ step: Fredholm setting.

Uses Melrose '93, radial point estimates Melrose '94, Vasy '13, non elliptic Fredholm framework Vasy '13.

Proposition

*Let $0 \leq a < \mathfrak{m}$. There exists $s_2 > 0$ with the following property.*

1. *Let $s > s_2$, $\ell < -1/2, s + \ell > -1/2$. Then the operators*

$$\widehat{L_b}(\sigma)\colon \{u \in \bar{H}_{\mathrm{b}}^{s,\ell}(X)\colon \widehat{L_{g_b}}(\sigma)u \in \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X)\} \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X)$$

   *are Fredholm operators of index $0$ for $\operatorname{Im}\sigma \geq 0$, $\sigma \neq 0$.*

2. *If moreover $\ell \in (-3/2, -1/2)$, then*

$$\widehat{L_b}(0)\colon \{u \in \bar{H}_{\mathrm{b}}^{s,\ell}(X)\colon \widehat{L_{g_b}}(0)u \in \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X)\} \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X)$$

   *is Fredholm of index $0$.*

## $2^{\text{nd}}$ step. Mode analysis for the gauged fixed linearized Einstein equations

- $\widehat{L_b}(\sigma)$ invertible for $\mathrm{Im}\sigma \geq 0$, $\sigma \neq 0$.
- Precise description of the kernel of $\widehat{L_b}(0)$, $\ell \in (-3/2, -1/2)$.

The mode analysis of $\widehat{L_b}(\sigma)$ follows from the mode analysis of

- ungauged linearized Einstein equations,
- the $1-$ form wave operator.

One also has to consider generalized mode solutions
$h = \sum_{j=0}^{d} t_*^j h_j$, $h_j \in \bar{H}_{\mathrm{b}}^{\infty, -1/2-}$, $L_{b_0} h = 0$.
There exist such solutions with $h_d \neq 0$, $d \geq 2$.

# Mode stability without gauge

Theorem (Andersson-H-Whiting '22)

*Let* $0 \leq a < \mathfrak{m}$, $\sigma \in \mathbb{C}$, $\operatorname{Im}\sigma \geq 0$, *and suppose* $\dot{g} = e^{-i\sigma t_*}h$, $h \in \bar{H}_{\mathrm{b}}^{\infty,\ell}(X; S^2\widetilde{{}^{\mathrm{sc}}T^*X})$, $\ell \in (-3/2, -1/2)$ *is an outgoing mode solution of the linearized Einstein equation*

$$D_g \mathrm{Ric}(\dot{g}) = 0.$$

*If* $\sigma = 0$, *then there exist parameters* $\dot{\mathfrak{m}} \in \mathbb{R}$, $\dot{\mathfrak{a}} \in \mathbb{R}^3$, *and a 1-form* $\omega \in \bar{H}_{\mathrm{b}}^{\infty,\ell-1}(X; \widetilde{{}^{\mathrm{sc}}T^*}X)$, *such that*

$$\dot{g} - \dot{g}_{(\mathfrak{m},\mathfrak{a})}(\dot{\mathfrak{m}}, \dot{\mathfrak{a}}) = \delta_g^* \omega.$$

*If* $\sigma \neq 0$, *then* $\dot{g} = \delta_g^* \omega$ *for a suitable outgoing* 1- *form* $\omega$.

# Mode stability with gauge

## Theorem (Andersson-H-Whiting '22)

*Let* $0 < a < \mathfrak{m}$.

1. *For* $\operatorname{Im}\sigma \geq 0$, $\sigma \neq 0$, *the operator*

$$\widehat{L_{g_b}}(\sigma)\colon \{u \in \bar{H}_{\mathrm{b}}^{s,\ell}(X; S^2\,\widetilde{^{\mathrm{sc}}T^*}X)\colon \widehat{L_{g_b}}(\sigma)u \in \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X; S^2\,\widetilde{^{\mathrm{sc}}T^*}X)\} \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X; S^2\,\widetilde{^{\mathrm{sc}}T^*}X)$$

*is invertible when* $s > s_2$, $\ell < -\frac{1}{2}$, $s + \ell > -\frac{1}{2}$.

2. *If moreover* $\ell \in (-\frac{3}{2}, -\frac{1}{2})$, *the zero energy operator*

$$\widehat{L_{g_b}}(0)\colon \{u \in \bar{H}_{\mathrm{b}}^{s,\ell}(X; S^2\,\widetilde{^{\mathrm{sc}}T^*}X)\colon \widehat{L_{g_b}}(0)u \in \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X; S^2\,\widetilde{^{\mathrm{sc}}T^*}X)\} \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}(X; S^2\,\widetilde{^{\mathrm{sc}}T^*}X)$$

*has 7-dimensional kernel and cokernel.*

# Mode analysis with and without gauge

$\sigma \neq 0$

$$L_b h = D_{g_b}\mathrm{Ric}(h) - \delta^*_{g_b} D_{g_b} W(h) = 0, \quad h = e^{-i\sigma t_*} h_0, \tag{1}$$

$h_0 = h_0(r, \omega)$ fulfills outgoing radiation condition. Apply linearized second Bianchi identity

$$\Box_{b,1}(D_{g_b} W(h)) = 0.$$

We show absence of modes for the 1− form wave operator :

$$D_{g_b} W(h) = 0. \tag{2}$$

Put into (1):

$$D_{g_b}\mathrm{Ric}(h) = 0.$$

$\Rightarrow$ $h$ is pure gauge $h = \delta^*_{g_b}\omega$. Putting this into (2) gives

$$\Box_{b,1}\omega = 0.$$

It follows $\omega = 0$.

# Proof of mode stability

$\sigma = 0$.

- The Teukolsky scalars are zero (Whiting).
- Gauge invariants. For linearized perturbations of Kerr with vanishing Teukolsky scalars, the only non vanishing gauge invariants are $\mathbb{I}_\xi$, $\mathbb{I}_\zeta$ and they have in this case exactly the form they have for the linearized Plebanski-Demianski family.
- The set of gauge invariants {Teukolsky scalars, linearized Ricci tensor, $\mathbb{I}_\xi, \mathbb{I}_\zeta$} is complete (Aksteiner, Andersson, Bäckdahl, Khavkine, Whiting '19), it follows that up to gauge the perturbation is a Plebanski-Demianski line element.
- For all $\dot{\mathfrak{a}} \in \mathbb{R}^3$, there exists $\lambda(\dot{\mathfrak{a}}) \in \mathbb{R}$, a 1−form $\omega \in \bar{H}_{\mathrm{b}}^{\infty,\ell-1}(X; \widetilde{{}^{\mathrm{sc}}T^*}X)$ and $g_0 \in C^\infty(\partial X; S^2 \widetilde{{}^{\mathrm{sc}}T^*_{\partial X}}X)$ such that

$$\dot{g} - \delta^*_{g_b}\omega - \dot{g}_b(\lambda(\dot{\mathfrak{a}}), \dot{\mathfrak{a}}) - \frac{1}{r} g_0 \in \mathcal{O}(r^{-2+}).$$

- Asymptotic behavior considerations eliminate the nut and acceleration parameters.

# $3^{rd}$ step: Constraint dumping

Gauge freedom (Constraint dumping)
Gundlach et al '05.
Zeroth order modification of $\delta_g^*$:

$$\tilde{\delta}_g^* = \delta_g^* + E,\ E\omega = \gamma(2c \otimes_s \omega - g^{-1}(c,\omega)g),$$

$c$ being a stationary $1-$ form with compact spatial support.
Replacing $\delta_g^*$ by $\tilde{\delta}_g^*$ gives a modified gauge fixed linear operator.
For $\gamma \neq 0$ small no quadratically growing generalized modes exist.
All other important properties of $L_b$ remain unchanged.

# $4^{th}$ step : Regularity of the resolvent at high frequencies

Proposition

*Let $\ell < -\frac{1}{2}$, and $s > \frac{5}{2}$, $s + \ell > -\frac{1}{2} + m$, $m \in \mathbb{N}$. Let $\sigma_0 > 0$. Then for $\operatorname{Im} \sigma \geq 0$, $|\sigma| > \sigma_0$, $h = |\sigma|^{-1}$, the operator*

$$\partial_\sigma^m \widehat{L_b}(\sigma)^{-1} : \bar{H}_{\mathrm{b},h}^{s,\ell+1} \to h^{-m} \bar{H}_{\mathrm{b},h}^{s-m,\ell}$$

*is uniformly bounded.*

Main issue at high frequencies ($|\mathrm{Re}\sigma| \to \infty$): Trapping, see Wunsch-Zworski '11, Dyatlov '16, Hintz '17.
Remark : the trapping is $r-$ normally hyperbolic for $0 \leq a < \mathfrak{m}$ (Dytalov '15), therefore the above proposition should hold for $0 \leq a < \mathfrak{m}$.

## $5^{th}$ step: Structure of the resolvent at low frequencies

- There exists $V \in \Psi^{-\infty}(X^\circ; S^2 T^* X^\circ)$,

$$\check{L}_b(\sigma) := \widehat{L_b}(\sigma) + V\colon \mathcal{X}_b^{s,\ell}(\sigma) \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}$$

  is invertible for $\sigma \in \mathbb{C}$, $\operatorname{Im}\sigma \geq 0$, $b$ close to $b_0$.
- Kernel of $\widehat{L_b}(0)$ :

$$\mathcal{K}_b = \mathcal{K}_{b,s} \oplus \mathcal{K}_{b,v}, \quad \tilde{\mathcal{K}}_{b,\circ} = \check{L}_b(0)\mathcal{K}_{b,\circ}(0), \ \circ = s, v.$$

- There exists $\Pi_b^\perp\colon \bar{H}_{\mathrm{b}}^{s-1,\ell+2} \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}$ of rank 7 which depends continuously on $b$ near $b_0$, and satisfies

$$\langle \Pi_b f, h^* \rangle = 0 \quad \forall\, h^* \in \mathcal{K}_b^*, \ \Pi_b = 1 - \Pi_b^\perp.$$

- Consider $\widehat{L_b}(\sigma)\check{L}_b(\sigma)^{-1} : \bar{H}_{\mathrm{b}}^{s-1,\ell+2} \to \bar{H}_{\mathrm{b}}^{s-1,\ell+2}$. Decomposition

$$\text{domain:} \quad \bar{H}_{\mathrm{b}}^{s-1,\ell+2} \cong \widetilde{\mathcal{K}}^\perp \oplus \widetilde{\mathcal{K}}_{b,s} \oplus \widetilde{\mathcal{K}}_{b,v},$$

$$\text{target:} \quad \bar{H}_{\mathrm{b}}^{s-1,\ell+2} \cong \operatorname{ran}\Pi_b \oplus \mathcal{R}_s^\perp \oplus \mathcal{R}_v^\perp.$$

$$\widehat{L_b}(\sigma)\check{L}_b(\sigma)^{-1} = \begin{pmatrix} L_{00} & \sigma\widetilde{L}_{01} & \sigma\widetilde{L}_{02} \\ \sigma\widetilde{L}_{10} & \sigma^2\widetilde{L}_{11} & \sigma^2\widetilde{L}_{12} \\ \sigma\widetilde{L}_{20} & \sigma^2\widetilde{L}_{21} & \sigma\widetilde{L}_{22} \end{pmatrix}.$$

We then obtain :

$$\widehat{L_b}(\sigma)^{-1} = \sigma^{-2}R_2 + \sigma^{-1}R_1 + L_b^-,$$

where the range of $R_1, R_2$ is explicit (linearized Kerr, pure gauge).
Regularity of $L_b^-$ (uses Vasy'19):

### Proposition

*Let $\ell \in (-\frac{3}{2}, -\frac{1}{2})$, $\epsilon \in (0,1)$, $\ell + \epsilon \in (-\frac{1}{2}, \frac{1}{2})$, and $s - \epsilon > \frac{7}{2}$. Then we have*

$$L_b^- \in H^{3/2-\epsilon}((-\sigma_0, \sigma_0); \mathcal{L}(\bar{H}_{\mathrm{b}}^{s-1,\ell+2}, \bar{H}_{\mathrm{b}}^{s-\max(\epsilon,1/2),\ell+\epsilon-1})).$$

Main issue : behavior of the metric at infinity.

Thank you for your attention !