Revisiting the Strong Cosmic Censorship for scalar field in Kerr interior

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Einstein's theory of General Relativity (1915)

$(\mathcal{M}, g_{\alpha\beta})$ a $1+3$ dimensional Lorentzian manifold with signature $(-,+,+,+)$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R \cdot g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where $R_{\alpha\beta}$ is Ricci tensor, R Ricci scalar, Λ cosmological constant, and $T_{\alpha\beta}$ stress-energy tensor.

The vacuum Einstein equation:

$$T_{\alpha\beta} = 0 \ \Rightarrow \ R_{\alpha\beta} = \Lambda g_{\alpha\beta}.$$

Second-order quasilinear PDE system for metric.

Explicit solutions to $R_{\alpha\beta} = 0$ or Einstein–Maxwell

1. **Minkowski**: $g = -(\mathrm{d}t)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2$.
2. **Schwarzschild** (1915): $g_M = -\mu \mathrm{d}t^2 + \mu^{-1}\mathrm{d}r^2 + r^2 \mathrm{d}\sigma_{S^2}$, $\mu = 1 - \frac{2M}{r}$. Spherically symmetric, asymptotically flat, contains a black hole;
3. **Kerr** (1963): subextremal family of axisymmetric, rotating spacetimes

$$g_{M,a} = -\left(1 - \frac{2Mr}{|q|^2}\right)\mathrm{d}t^2 - \frac{2aMr\sin^2\theta}{|q|^2}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) + \frac{|q|^2}{\Delta}\mathrm{d}r^2 + |q|^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{|q|^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2,$$

   with $0 \leq |a| < M$, $\Delta(r) = r^2 - 2Mr + a^2$, $|q|^2 = r^2 + a^2\cos^2\theta$.
4. **Reissner–Nordström** for Einstein–Maxwell: $g_{M,e} = -\mu\mathrm{d}t^2 + \mu^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\sigma_{S^2}$, where $\mu = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, Q is the charge of the black hole and satisfies $|Q| < M$.
5. **Kerr–Newman** for Einstein–Maxwell

Maximal Cauchy development and SCC

Maximal Cauchy development (Choquet-Bruhat and Geroch 1969)

There exists a maximal Cauchy (or globally hyperbolic) development of the Cauchy data $(\bar{M}, \bar{g}, \bar{K})$ with $\bar{g} \in H^s$, $\bar{K} \in H^{s-1}$, $s > \frac{n}{2} + 1$, which is unique up to isometry.

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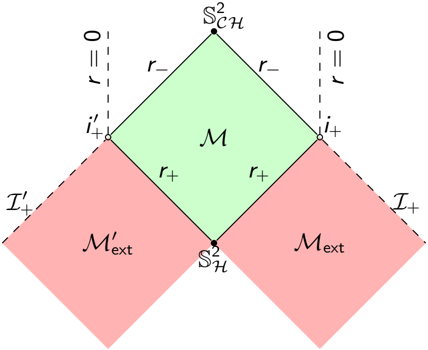
Strong Cosmic Censorship hypothesis, Penrose

For generic vacuum, asymptotically flat initial data, the maximal Cauchy development is inextendible as a suitably regular Lorenztian manifold.

Motivation of SCC: Kerr spacetimes



There are infinitely many smooth extensions beyond the Cauchy horizon of Kerr (similarly for RN).

SCC hypothesis about Kerr (or RN)

For sufficiently small perturbations of Kerr (or RN) initial data, the Cauchy horizon is generically inextendible (thus, unstable) in a suitable regularity (say, H^1_{loc}) sense.

Literature on SCC for Einstein(-coupled) equation

- Physics literature: Poisson–Israel, Ori, McNamara, $\cdots$
- Weak null singularity (Christoffel symbols blow up and are not square integrable): Luk (17')
- Spherically symmetric Einstein-Maxwell-(real) scalar: Dafermos (03') and Dafermos–Rodnianski (05') for C^0-extendibility and mass inflation; Luk–Oh (19') for C^2-inextendibility and Sbierski (20') for $C^{0,1}_{\mathrm{loc}}$-inextendibility; Costa–Girao–Natario–Silva (17',18') with a cosmological constant
- Spherically symmetric Einstein-Maxwell-charged (massive) scalar field: Van de Moortel (18', 21') proved C^0-extendibility and C^2-inextendibility under assumptions on the decay for the massive scalar field on event horizon.
- Dafermos–Luk (17'): C^0-stability of the Kerr CH assuming Kerr stability

Literature on SCC for simplified models

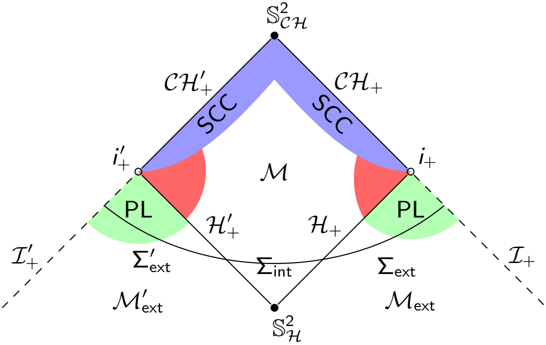
Scalar field on RN and Kerr

- C^0-extendibility: Franzen (16') on RN; Hintz (17') and Franzen (20') on Kerr
- H^1_{loc}-inextendibility: Sbierski (15') on Kerr using Gaussian beam approximation; Luk–Oh (15') on RN; Luk–Sbierski (16') on Kerr; Luk–Oh–Shlapentokh-Rothman (22') on RN by the scattering map near 0 time-frequency

Spin-2 Teukolsky on Kerr

Sbierski (22') showed the instability of Kerr CH for the Teukolsky equation for spin $+2$ component by the scattering map near 0 time-frequency

Global asymptotics of linear perturbations



Cauchy problem for scalar field or linearized gravity in Kerr spacetimes.

- Upper and lower bounds of decay in the exterior region: Price's law
- Asymptotics near event horizon
- Asymptotics near Cauchy horizon (SCC)

Spin s components

In the Newman–Penrose formalism, choose at each point a complex null tetrad $(l, n, m, \bar{m})$ s.t. $g(l, n) = -1$, $g(m, \bar{m}) = 1$ and the other products being zero.

The spin s components, $s = 0, \pm 1, \pm 2$, are

$$\Upsilon_{+1} = \mathrm{F}_{lm}, \qquad \Upsilon_{-1} = \mathrm{F}_{\bar{m}n},$$
$$\Upsilon_{+2} = \mathrm{W}_{lmlm}, \qquad \Upsilon_{-2} = \mathrm{W}_{n\bar{m}n\bar{m}}.$$

Hartle–Hawking tetrad (n is geodetic $\nabla_n n = 0$, l and n are principal null)

$$l^\nu = \frac{1}{\sqrt{2}\Sigma}(r^2 + a^2, \Delta, 0, a), \qquad n^\nu = \frac{1}{\sqrt{2}\Delta}(r^2 + a^2, -\Delta, 0, a),$$
$$m^\nu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}(ia\sin\theta, 0, 1, \frac{i}{\sin\theta}), \qquad \bar{m}^\nu = \text{c.c. of } m^\nu.$$

Teukolsky Master Equation (TME, '72)

TME for spin s components

Let $\mathfrak{s} = |s| \in \{0, \frac{1}{2}, 1, 2\}$. The spin $s = \pm\mathfrak{s}$ components

$$\psi_{+\mathfrak{s}} \doteq |q|^{2\mathfrak{s}} \Upsilon_{+\mathfrak{s}} \approx r^{2\mathfrak{s}} \Upsilon_{+\mathfrak{s}}, \quad \psi_{-\mathfrak{s}} \doteq |q|^{-2\mathfrak{s}} (r - ia\cos\theta)^{2\mathfrak{s}} \Upsilon_{-\mathfrak{s}} \approx \Upsilon_{-\mathfrak{s}}$$

solve a *decoupled*, *separable* spin-weighted wave eq:

$$0 = |\mathrm{q}|^2 \Box_{\mathrm{g}} \psi_{\mathrm{s}} + \frac{2\mathrm{is}\cos\theta}{\sin^2\theta} \partial_\phi \psi_{\mathrm{s}} - (\mathrm{s}^2 \cot^2\theta + \mathrm{s})\psi_{\mathrm{s}} - 2\mathrm{ias}\cos\theta \partial_{\mathrm{t}} \psi_{\mathrm{s}} \rightarrow |q|^2 \Box_{g,s} \psi_s$$
$$- 2s\left(\frac{r^3 - 3Mr^2 + a^2 r + a^2 M}{\Delta} \partial_t + (r - M)\partial_r - \frac{a(r-M)}{\Delta} \partial_\phi \right) \psi_s.$$

$\{\psi_s\}_{s=\pm\mathfrak{s}}$ govern the dynamics of scalar, Dirac, Maxwell and linearized gravity.

Price's Law: a law on the generically sharp decay rates for linear models

Price's law for spin fields on Schw, RN, Kerr

1. On Schwarzschild. (and RN):

	towards null infinity	finite radius region
$r^{-\mathfrak{s}-s}\psi_s \approx \Upsilon_s$	$r^{-1-\mathfrak{s}-s}u^{-2-\mathfrak{s}+s}$	$\underline{u}^{-3-2\mathfrak{s}}$
$(r^{-\mathfrak{s}-s}\psi_s)_\ell \approx (\Upsilon_s)_\ell$	$r^{-1-\mathfrak{s}-s}u^{-2-\ell+s}$	$\underline{u}^{-3-2\ell}$

2. In Kerr, $r^{-\mathfrak{s}-s}\psi_s \simeq \underline{u}^{-1-\mathfrak{s}-s}\tau^{-2-\mathfrak{s}+s}$. In a finite region of Kerr: for scalar field, $(\psi)_{\geq \ell}$ has decay $\underline{u}^{-3-\ell}$ for even ℓ and $\underline{u}^{-4-\ell}$ for odd ℓ; for $\mathfrak{s} \neq 0$, $(r^{-\mathfrak{s}-s}\psi_s)_{\geq \ell}$ has decay $\underline{u}^{-3-\mathfrak{s}-\ell}$.

- Donninger–Schlag–Soffer (11', 12'): *Locally*, $t^{-2\ell-2}$ for a fixed ℓ mode for a Regge–Wheeler eq, t^{-3} scalar, t^{-4} Maxwell, t^{-6} for linearized gravity on Schw.
- Metcalfe–Tataru–Tohaneanu (12', 13', 17'): sharp decay for scalar and $\underline{u}^{-2+|s|}\tau^{-2+|s|}$ decay for Maxwell in a class of non-stationary AFST under assumptions
- Hintz (20'): For scalar, **PL** on Kerr and $\geq \ell$ modes on Schw.
- Angelopoulos–Aretakis–Gajic (18',21'): **PL** for scalar field and its $\geq \ell$ modes on RN, and for $\ell = 0$, $\ell = 1$ and $\ell \geq 2$ modes of scalar field on Kerr
- Ma (20'), Ma–Zhang (20',21'): **PL** for Dirac on Schw., **PL** for spin-s fields and their $\geq \ell$ modes on Schw, and **PL** for spin-s fields on Kerr

Precise asymptotics for the scalar field on the Kerr (or RN) event horizon

Theorem 1 (Hintz; Angelopoulos-Aretakis-Gajic; Ma-Zhang)

For scalar field ψ arising from smooth, compactly supported initial data on a two-ended hypersurface Σ_{init}, then

$$\psi = c_0 \underline{u}^{-3} + O(\underline{u}^{-3-\epsilon}), \quad \text{on} \quad \mathcal{H}_+ \tag{1}$$

where

$$c_0 = -\frac{2M}{\pi}\left((r_+^2 + a^2)\int_{\Sigma_{ext}\cap\mathcal{H}} \psi \mathrm{d}\omega - \int_{\Sigma_{ext}} |q|^2 \langle \nabla\tau, \nabla\psi\rangle_{g_{M,a}} \mathrm{d}\rho \mathrm{d}\omega\right). \tag{2}$$

Further, for generic such initial data, the constant c_0 is non-zero.

Theorem 2 (Angelopoulos-Aretakis-Gajic)

Furthermore, $\psi_{\ell\geq 1} \sim \underline{u}^{-5}$ on $\mathcal{H}_+$.

Remark

These estimates are also valid slightly inside the black hole.

The Kerr black hole interior region

Let $u = r^* - t$ and $\underline{u} = r^* + t$, with $dr^* = \mu^{-1}dr$, $\mu = \frac{\Delta}{r^2+a^2}$. Let $\gamma_0 \in (0, \frac{1}{2})$.

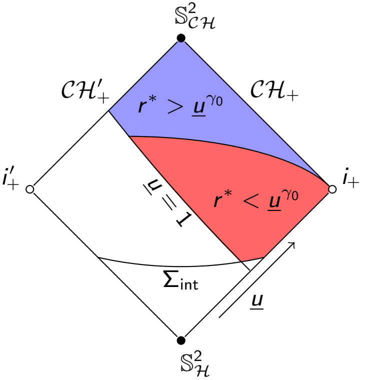


Figure: Region ${}_r\mathcal{D}^+_{\text{init}}$ and its subregions

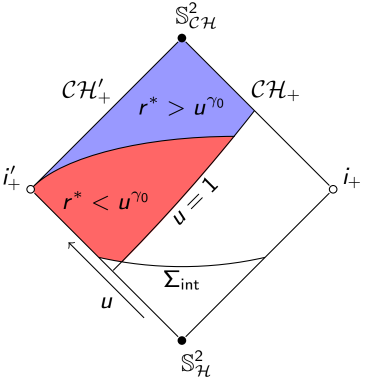


Figure: Region ${}_l\mathcal{D}^+_{\text{init}}$ and its subregions

Precise asymptotics for the scalar field in the interior of Kerr BH

Theorem 3 (Ma-Zhang 22': on solution itself)

Assume $\psi = c_0 \underline{u}^{-3} + O(\underline{u}^{-3-\epsilon})$ *and* $|\psi_{\ell\geq 1}| \lesssim \underline{u}^{-4-\delta}$ *on event horizon,* $\delta > 0$. *Then there exists a smooth function* $\Psi(u,\omega)$, ω *being the spherical coordinates on* $\mathbb{S}^2_{u,\underline{u}}$, *such that*

$$|\psi - c_0 \underline{u}^{-3}| \lesssim \underline{u}^{-3-\epsilon} \quad \text{in } {}_r\mathcal{D}^+_{init} \cap \{r^* \leq \underline{u}^{\gamma_0}\}, \tag{3a}$$

$$\left|\psi - \Psi(u,\omega) - \frac{1}{2}c_0\left(1 + \frac{r_+^2 + a^2}{r^2 + a^2}\right)\underline{u}^{-3}\right| \lesssim \underline{u}^{-3-\epsilon} \quad \text{in } {}_r\mathcal{D}^+_{init} \cap \{r^* \geq \underline{u}^{\gamma_0}\}, \tag{3b}$$

where $\gamma_0 \in (0, \frac{1}{2})$ *is an arbitrary constant and*

$$\left|\Psi(u,\omega) + \frac{1}{2}c_0\left(1 - \frac{r_+^2 + a^2}{r_-^2 + a^2}\right)u^{-3}\right| \lesssim |u|^{-3-\epsilon} \quad \text{as } u \to -\infty, \tag{3c}$$

$$\left|\Psi(u,\omega) - \frac{1}{2}c_0'\left(1 + \frac{r_+^2 + a^2}{r_-^2 + a^2}\right)u^{-3}\right| \lesssim |u|^{-3-\epsilon} \quad \text{as } u \to +\infty; \tag{3d}$$

Define two principal null directions

$$e_3 \doteq \frac{1}{2}\left(\frac{r^2+a^2}{\Delta}\partial_t + \frac{a}{\Delta}\partial_\phi - \partial_r\right), \qquad e_4 \doteq \frac{1}{2}\left(\partial_t + \frac{a}{r^2+a^2}\partial_\phi + \frac{\Delta}{r^2+a^2}\partial_r\right).$$

Also, define $e_3' \doteq (-\mu)e_3$ and $e_4' \doteq (-\mu)^{-1}e_4$.

Theorem 4 (Ma-Zhang 22': on the derivatives of the solution)

For $e_4\psi$, we have

$$\left|e_4\psi + \frac{3}{2}c_0\left(1+\frac{r_+^2+a^2}{r^2+a^2}\right)\underline{u}^{-4}\right| \lesssim \underline{u}^{-4-\epsilon} \quad \text{in } {}_r\mathcal{D}^+_{init}.$$

For $(-\mu e_3)\psi$, we have

$$\begin{aligned}
&\left|(-\mu e_3)\psi - \frac{3}{2}c_0\left(1-\frac{r_+^2+a^2}{r^2+a^2}\right)\underline{u}^{-4}\right| \lesssim (r_+-r)\underline{u}^{-4-\epsilon} && \text{in } {}_r\mathcal{D}^+_{init}\cap\{r^*\le \underline{u}^{\gamma_0}\},\\
&\left|(-\mu e_3)\psi - (-\mu e_3)|_{\mathcal{CH}_+}(\Psi(u,\omega))\right| \lesssim -\mu && \text{in } {}_r\mathcal{D}^+_{init}\cap\{r^*\ge 0\},
\end{aligned}$$

where

$$\left|(-\mu e_3)|_{\mathcal{CH}_+}(\Psi(u,\omega)) - \frac{3}{2}c_0\left(1-\frac{r_+^2+a^2}{r_-^2+a^2}\right)u^{-4}\right| \lesssim |u|^{-4-\epsilon} \quad \text{as } u\to-\infty.$$

A few more comments

Also, $|e_3\psi + \frac{3}{2}c_0\frac{r+r_+}{r-r_-}\underline{u}^{-4}| \lesssim \underline{u}^{-4-\epsilon}$ in ${}_r\mathcal{D}^+_{\text{init}} \cap \{r_0 \le r \le r_+\}$ for any given $r_0 \in (r_-, r_+)$.

Estimates in the left of black hole interior

Meanwhile, there exists a smooth function $\Psi'(\underline{u}, \omega)$ such that the above estimates are valid in ${}_l\mathcal{D}^+_{\text{init}} \doteq \mathcal{D}^+_{\text{init}} \cap \{u \ge 1\}$ if we make the replacements $u \to \underline{u}$, $\underline{u} \to u$, $e_3 \to e_4' = (-\mu)^{-1}e_4$, $e_4 \to e_3' = -\mu e_3$, $\Psi(u,\omega) \to \Psi'(\underline{u},\omega)$, ${}_r\mathcal{D}^+_{\text{init}} \to {}_l\mathcal{D}^+_{\text{init}}$, $\{r^* \le \underline{u}^{\gamma_0}\} \to \{r^* \le u^{\gamma_0}\}$, $\mathcal{CH}_+ \to \mathcal{CH}'_+$, respectively.

The estimates are invariant under $T = \partial_t$ operation on both sides.

Globality of the estimates in black hole interior

since the remaining region $\mathcal{D}^+_{\text{init}} \cap \{\underline{u} \le 1\} \cap \{u \le 1\}$ is a compact region with both u and $\underline{u}$ uniformly bounded from above and below.

Estimates in RN hold as well by let $a = 0$, $\Delta = r^2 - 2Mr + Q^2$, and $\mu = \frac{\Delta}{r^2} = \frac{r^2-2Mr+Q^2}{r^2}$.

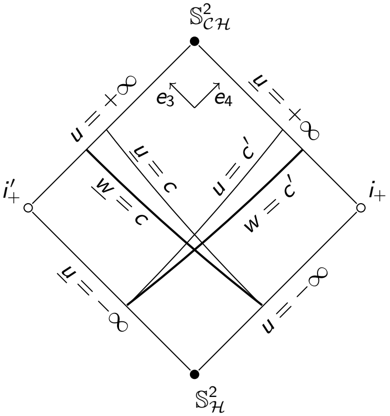
H^1_{loc}-inextendibility

Define $w \doteq u - r + r_-$, $\underline{w} \doteq \underline{u} - r + r_+$. The constant-$w$ and -$\underline{w}$ hypersurfaces $\mathcal{C}_w$ and $\underline{\mathcal{C}}_{\underline{w}}$ are spacelike.

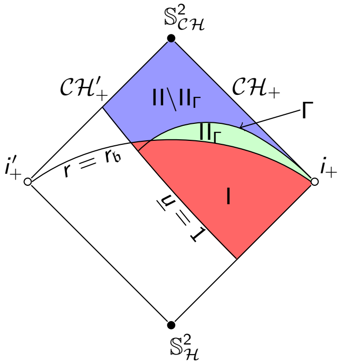
H^1_{loc} inextendibility

Since c_0 is generically non-zero, we conclude

1. The regular derivative $(-\mu)^{-1} e_4 \psi$ generically blows up towards the right Cauchy horizon $\mathcal{CH}_+$.
2. The nondegenerate energy of ψ on hypersurface $\mathcal{C}_w \cap \{\underline{u} \geq \underline{u}_0\}$, which bounds $\int_{\mathcal{C}_w \cap \{\underline{u} \geq \underline{u}_0\}} |\mu|^{-1} |e_4 \psi|^2 \mathrm{d}\underline{u}$, generically goes to $+\infty$ as $\underline{u}_0 \to +\infty$.
3. One can examine the validity of SCC in a weak regularity space.



Sketch of the proof



- I: red-shift region
- II_Γ: $\underline{u} \sim \underline{u}_{r_\mathfrak{b}}(u)$
- $\mathrm{II}\backslash\mathrm{II}_\Gamma$: blue-shift region

$\psi = \psi_{\ell=0} + \psi_{\ell\geq 1}$, spherically symmetric part and the remaining part

Integrate along $\underline{u} = const$ starting from event horizon + fast energy decay

The equation satisfied by $\psi_{\ell=0}$:

$$\partial_u\left((r^2+a^2)\partial_{\underline{u}}\psi_{\ell=0} - \frac{1}{2}(r^2+a^2)T\psi_{\ell=0}\right)$$
$$= -\frac{1}{2}(r^2+a^2)\partial_u T\psi_{\ell=0} - \frac{1}{4}a^2\mu\mathbb{P}_{\ell=0}(\sin^2\theta T^2\psi)$$

Sketch of the proof (energy decay estimates)

Region I: red-shift estimate

$$\int_{\underline{u}=const} (-\mu)^{-1} |\partial_u T^j \psi_{\ell=0}|^2 \mathrm{d}u \lesssim \underline{u}^{-8-2j},$$

Region II_Γ: blue-shift estimate

$$\int_{\underline{u}=const} |\log(-\mu)|^{-\frac{1}{2}} |\partial_u T^j \psi_{\ell=0}|^2 \mathrm{d}u \lesssim \underline{u}^{-8-2j+\gamma},$$

Region $\mathrm{II} \backslash \mathrm{II}_\Gamma$: we only need boundedness,

$$\int_{\underline{u}=const} |\log(-\mu)|^{-\frac{5}{2}} |T^j \psi_{\ell=0}|^2 \mathrm{d}u \lesssim 1$$

Since $-\mu$ has exponential decay in this region, the error terms are easily controlled.

Thank you!