Linear and non-linear stability of collisionless many-particle systems on black hole exteriors

Renato Velozo Ruiz

Sorbonne Université

Seminar on Mathematical General Relativity
Laboratoire Jacques-Louis Lions
Sorbonne Université
December 15, 2022

Relativistic collisionless many-particle systems

This work is motivated by the dynamics of the solutions $(\mathcal{M}^{1+3}, g, f)$ to the *Einstein–Vlasov system* (EV)

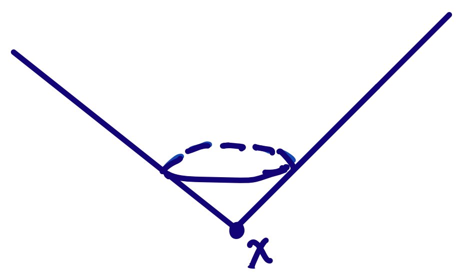
$$\begin{cases} \mathrm{Ric}(g)_{\mu\nu} - \frac{1}{2}\mathrm{R}(g)g_{\mu\nu} = 8\pi \mathrm{T}_{\mu\nu}(f), \\ \mathrm{T}_{\mu\nu}(f) := \int_{\mathcal{P}_x} f p_\mu p_\nu \,\mathrm{dvol}(p), \\ X(f) = 0. \end{cases}$$

The distribution function $f(x,p)$ satisfies the *relativistic Vlasov equation* $X(f) = 0$ on the set

$$\mathcal{P}_\sigma := \Big\{ (x,p) \in T\mathcal{M} : g_x(p,p) = -\sigma^2, \text{ where } p \text{ is future directed} \Big\}.$$

i) $\sigma = 0$ massless

ii) $\sigma = 1$ massive



$\mathcal{P}_x$

Relativistic collisionless many-particle systems

This work is motivated by the dynamics of the solutions $(\mathcal{M}^{1+3}, g, f)$ to the *Einstein–Vlasov system* (EV)

$$\begin{cases} \mathrm{Ric}(g)_{\mu\nu} - \frac{1}{2}\mathrm{R}(g)g_{\mu\nu} = 8\pi \mathrm{T}_{\mu\nu}(f), \\ \mathrm{T}_{\mu\nu}(f) := \int_{\mathcal{P}_x} f p_\mu p_\nu \,\mathrm{dvol}(p), \\ X(f) = 0. \end{cases}$$

The distribution function $f(x,p)$ satisfies the *relativistic Vlasov equation* $X(f) = 0$ on the set

$$\mathcal{P}_\sigma := \Big\{ (x,p) \in T\mathcal{M} : g_x(p,p) = -\sigma^2, \text{ where } p \text{ is future directed} \Big\}.$$

Theorem (Choquet-Bruhat 1971)

The Einstein–Vlasov system is locally well-posed in Sobolev regularity.

The Schwarzschild solution

ℛ

The simplest black hole spacetime is the so-called *Schwarzschild solution* (Schw, g_s), which solves the *Einstein vacuum equations* given by

★and EV!

$$\mathrm{Ric}(g)_{\mu\nu} = 0.$$

The exterior of Schwarzschild spacetime is described by the Lorentzian metric

$$g_s = -D(r)dt \otimes dt + \frac{1}{D(r)} dr \otimes dr + r^2 d\gamma_{\mathbb{S}^2}, \qquad D(r) := 1 - \frac{2M}{r},$$

where $t \in \mathbb{R}$, $r \in (2M, \infty)$, and $d\gamma_{\mathbb{S}^2}$ is the standard metric on the unit sphere $\mathbb{S}^2$.

1. Sph symm Einstein-massless Vlasov system

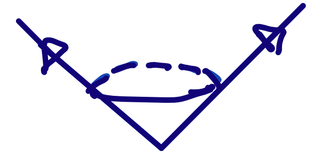
2. Massive Vlasov eqn on Schwarzschild

The Einstein–Vlasov system under spherical symmetry

Let $(\mathcal{M}^{3+1}, g)$ be a *spherically symmetric spacetime* in double null coordinates given by

*Charact. IVP.

$$g = -2\Omega^2(du \otimes dv + dv \otimes du) + r^2(u,v)d\gamma_{\mathbb{S}^2}.$$



We introduce the *spherically symmetric Einstein–Vlasov system* by

*Birkhoff.

$$\begin{cases} \partial_u\partial_v r &= -\frac{\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + 4\pi r \mathrm{T}_{uv}, \\ \partial_u\partial_v \log\Omega &= \frac{\Omega^2}{4r^2} + \frac{\partial_u r \partial_v r}{r^2} - 4\pi \mathrm{T}_{uv} - 4\pi\Omega^2 g^{AB}\mathrm{T}_{AB}, \\ \partial_u(\Omega^{-2}\partial_u r) &= -4\pi r \mathrm{T}_{uu}\Omega^{-2}, \\ \partial_v(\Omega^{-2}\partial_v r) &= -4\pi r \mathrm{T}_{vv}\Omega^{-2}, \\ X(f) &= 0, \end{cases}$$

where T_{AB}, T_{uu}, T_{uv} and T_{vv} are components of the energy momentum tensor.

Previous results – On massless Vlasov

1. Stability of Minkowski for the spherically symmetric Einstein–massless Vlasov system (Dafermos 2006).

2. Stability of Minkowski for the full Einstein–massless Vlasov system (Taylor 2017, Bigorgne–Fajman–Joudioux–Smulevici–Thaller 2021).

Previous results – On massless Vlasov

1. Stability of Minkowski for the spherically symmetric Einstein–massless Vlasov system (Dafermos 2006).

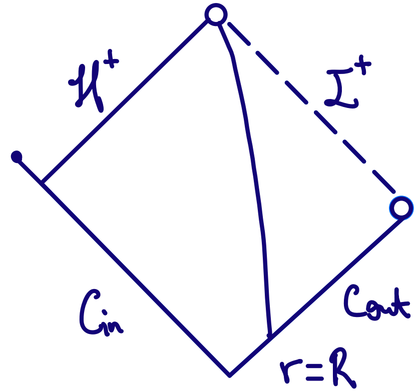
2. Stability of Minkowski for the full Einstein–massless Vlasov system (Taylor 2017, Bigorgne–Fajman–Joudioux–Smulevici–Thaller 2021).

3. Degenerated integrated energy estimate for the massless Vlasov equation on slowly rotating Kerr (Andersson–Blue–Joudioux 2018).

4. Decay for the massless Vlasov equation on Schwarzschild (Bigorgne 2020, Weissenbacher).

$T_{\mu\nu}(f)$

Asymptotic stability of Schwarzschild spacetime



Theorem (V.R. 2022)

For all initial data for (sEV) close to Schwarzschild with mass M_{in} with initial distribution function compactly supported, the resulting solution

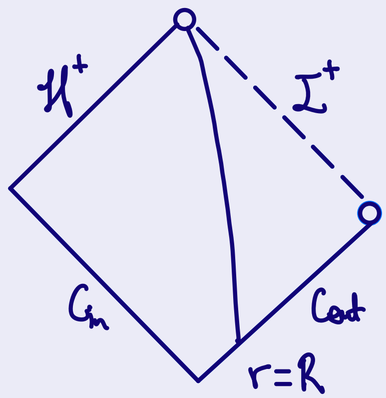
1. *possesses a complete future null infinity $\mathcal{I}^+$ whose past $J^-(\mathcal{I}^+)$ is bounded to the future by a complete event horizon $\mathcal{H}^+$;*
2. *remains close to the Schwarzschild solution with mass M_{in} in the exterior region;*
3. *has a metric g that asymptotes, inverse polynomially, to g_s with mass M_{fin};*
4. *has a matter content $\mathrm{T}_{\mu\nu}$ that decays* exponentially* to zero.* *continuity argument.

Decay for the massless Vlasov equation on Schwarzschild

Theorem (V.R. 2022)

Let f_0 be a compactly supported initial data for (mV). Let $\delta > 0$, and $R > 2M$ be sufficiently large. Then, the stress energy momentum tensor for the solution f of (mV) satisfies

$$\mathrm{T}_{uv} \leq \frac{C_1}{r^4 \exp((\frac{2}{3\sqrt{3}M} - \delta)u)},$$



for every $x \in \{r \geq R\}$, and

$$\mathrm{T}_{uv} \leq \frac{C_2(1 - \frac{2M}{r})}{\exp((\frac{2}{3\sqrt{3}M} - \delta)v)},$$

for every $x \in \{r \leq R\}$, where C_1 and C_2 are positive constants depending on f_0, δ, R, and M.

The geodesic flow in spherically symmetric spacetimes I

The geodesic equations are given by

$$\begin{cases} \frac{dp^u}{ds} & = -\partial_u \log \Omega^2 (p^u)^2 - \frac{\partial_v r}{\Omega^2}\frac{l^2}{2r^3}, \\ \frac{dp^v}{ds} & = -\partial_v \log \Omega^2 (p^v)^2 - \frac{\partial_u r}{\Omega^2}\frac{l^2}{2r^3}, \\ \frac{dp^\theta}{ds} & = -\frac{2p^r}{r}p^\theta + \sin\theta\cos\theta (p^\phi)^2, \\ \frac{dp^\phi}{ds} & = -\frac{2p^r}{r}p^\phi - 2\cot\theta p^\theta p^\phi. \end{cases}$$

We recall the standard *particle energy* E, the *angular momentum* l, and the *azimuthal angular momentum* l_ϕ along a geodesic given by

$$E := -\partial_u r p^u + \partial_v r p^v, \quad l := r^2\sqrt{(p^\theta)^2 + \sin^2\theta (p^\phi)^2}, \quad l_\phi := r^2 \sin^2\theta p^\phi.$$

Proposition

In a spherically symmetric spacetime $(\mathcal{M}, g)$, the quantities l and l_ϕ are conserved along the geodesic flow.

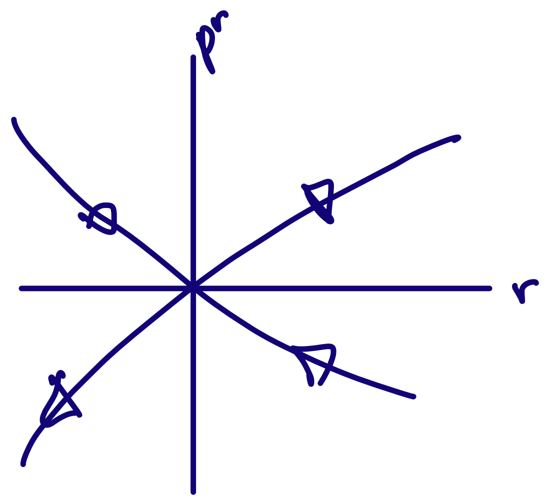
The geodesic flow in spherically symmetric spacetimes II

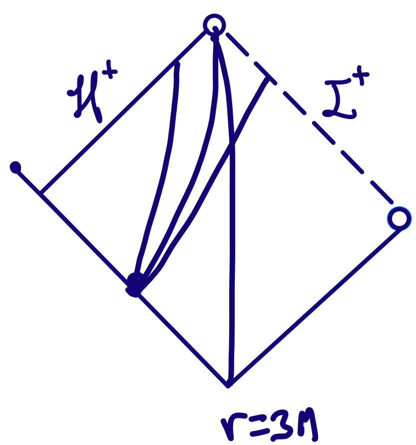
The geodesic equation for the area radius is given by

$$\frac{dp^r}{ds} = \frac{l^2}{r^4}(r - 3m) - 4\pi r\Big(\mathrm{T}_{uu}(p^u)^2 - 2\mathrm{T}_{uv}p^u p^v + \mathrm{T}_{vv}(p^v)^2\Big),$$

where the particle energy can be written as

$$E^2 = (p^r)^2 + \frac{l^2}{r^2}\Big(1 - \frac{2m}{r}\Big).$$





$$T_{\mu\nu} = \int_{\mathcal{P}_x} f\, p_\mu p_\nu \, d\mathrm{vol}(p)$$

The geodesic flow in spherically symmetric spacetimes II

The geodesic equation for the area radius is given by

$$\frac{dp^r}{ds} = \frac{l^2}{r^4}(r-3m) - 4\pi r\Big(\mathrm{T}_{uu}(p^u)^2 - 2\mathrm{T}_{uv}p^u p^v + \mathrm{T}_{vv}(p^v)^2\Big),$$

where the particle energy can be written as

$$E^2 = (p^r)^2 + \frac{l^2}{r^2}\Big(1 - \frac{2m}{r}\Big).$$

The normalized angular momentum can be written as

$$\frac{l^2}{E^2} - 27m^2 = \frac{r+6m}{r-2m}(r-3m)^2 - \frac{r^2}{D^3}\Big(\frac{Dp^r}{E}\Big)^2 = \varphi_+\varphi_-.$$

Moreover, the derivative of $\varphi_\pm$ along the geodesic flow is given by

$$\frac{d}{dt}\varphi_\pm = \pm\frac{1}{r^{1/2}(r+6m)^{1/2}}\varphi_\pm + \mathrm{Err}.$$

The geodesic flow in a neighborhood of the trapped set

Proposition

Under the bootstrap assumptions, there exist $\epsilon_0 > 0$ *and* $C > 0$ *such that for every* $(x, p) \in \operatorname{supp}(f) \cap \{|r - 3m| < \epsilon_0\}$ *for which the corresponding geodesic* γ *has normalized angular momentum* $\frac{l}{E} - 3\sqrt{3}m \in (-\epsilon_0, \epsilon_0)$, *we have*

$$\Big|\frac{Dp^r}{E} + \frac{(r-2m)\sqrt{r+6m}}{r^{\frac{5}{2}}}(r-3m)\Big| \leq \frac{C}{\exp\left(\left(\frac{2}{3\sqrt{3}m} - \frac{\delta}{2}\right)v\right)}.$$

This proposition is obtained using the stable manifold theorem for dynamical systems with a normally hyperbolic trapped set by Hintz 2021 after Hirsch–Pugh–Shub 1977.

The Sasaki metric

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. We recall the geometric decomposition

$$T_{(x,p)}T\mathcal{M} = \mathcal{H}_{(x,p)} \oplus \mathcal{V}_{(x,p)},$$

defined by the *horizontal lift* $\mathrm{Hor}_{(x,p)} : T_x\mathcal{M} \to T_{(x,p)}T\mathcal{M}$ and the *vertical lift* $\mathrm{Ver}_{(x,p)} : T_x\mathcal{M} \to T_{(x,p)}T\mathcal{M}$ given by

$$\mathrm{Hor}_{(x,p)}(Y^\alpha \partial_{x^\alpha}) := Y^\alpha \partial_{x^\alpha} - Y^\alpha p^\beta \Gamma^\gamma_{\alpha\beta} \partial_{p^\gamma}, \qquad \mathrm{Ver}_{(x,p)}(Y^\alpha \partial_{x^\alpha}) := Y^\alpha \partial_{p^\alpha}.$$

The Sasaki metric

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. We recall the geometric decomposition

$$T_{(x,p)}T\mathcal{M} = \mathcal{H}_{(x,p)} \oplus \mathcal{V}_{(x,p)},$$

defined by the *horizontal lift* $\mathrm{Hor}_{(x,p)} : T_x\mathcal{M} \to T_{(x,p)}T\mathcal{M}$ and the *vertical lift* $\mathrm{Ver}_{(x,p)} : T_x\mathcal{M} \to T_{(x,p)}T\mathcal{M}$ given by

$$\mathrm{Hor}_{(x,p)}(Y^\alpha \partial_{x^\alpha}) := Y^\alpha \partial_{x^\alpha} - Y^\alpha p^\beta \Gamma^\gamma_{\alpha\beta} \partial_{p^\gamma}, \qquad \mathrm{Ver}_{(x,p)}(Y^\alpha \partial_{x^\alpha}) := Y^\alpha \partial_{p^\alpha}.$$

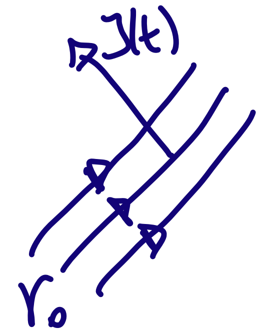
We define the *Sasaki metric* $\bar{g}$ on the tangent bundle $T\mathcal{M}$ by

$$\begin{aligned}
\bar{g}_{(x,p)}(\mathrm{Hor}_{(x,p)}(Y), \mathrm{Hor}_{(x,v)}(Z)) &= g_x(Y, Z), \\
\bar{g}_{(x,p)}(\mathrm{Hor}_{(x,p)}(Y), \mathrm{Ver}_{(x,v)}(Z)) &= 0, \\
\bar{g}_{(x,p)}(\mathrm{Ver}_{(x,p)}(Y), \mathrm{Ver}_{(x,v)}(Z)) &= g_x(Y, Z),
\end{aligned}$$

for every $(x, p) \in T\mathcal{M}$, and every $Y, Z \in T_x\mathcal{M}$.

Jacobi fields along the geodesic flow I

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Let $\epsilon > 0$. Let $\gamma_\tau : I \to \mathcal{M}$ be a one parameter family of geodesics where $\tau \in (-\epsilon, \epsilon)$, and $\gamma := \gamma_0$. A vector field $J(t) \in T_{\gamma(t)}\mathcal{M}$ given by



$$J(t) := \frac{\partial \gamma_\tau}{\partial \tau}(t)\Big|_{\tau=0}$$

is said to be a *Jacobi field on* $(\mathcal{M}, g)$ along the geodesic γ. Consequently, a Jacobi field J along γ satisfies the so-called *Jacobi equation*

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J = R(\dot{\gamma}, J)\dot{\gamma}.$$

Jacobi fields along the geodesic flow I

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Let $\epsilon > 0$. Let $\bar{\gamma}_\tau : I \to T\mathcal{M}$ be a one parameter family of geodesics on $T\mathcal{M}$ where $\tau \in (-\epsilon, \epsilon)$, and $\bar{\gamma} := \bar{\gamma}_0$. A vector field $\bar{J}(t) \in T_{\bar{\gamma}(t)}T\mathcal{M}$ given by

$$\bar{J}(t) := \frac{\partial \bar{\gamma}_\tau}{\partial \tau}(t)\Big|_{\tau=0},$$

is said to be a *Jacobi field on* $T\mathcal{M}$ along the geodesic $\bar{\gamma}$. Consequently, a Jacobi field $\bar{J}$ along γ satisfies the so-called *Jacobi equation*

$$\overline{\nabla}_X \overline{\nabla}_X \bar{J} = \bar{R}(X, \bar{J})X.$$

$J(t) = d\phi_t(x,p)(V)$

Jacobi fields along the geodesic flow I

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Let $\epsilon > 0$. Let $\bar{\gamma}_\tau : I \to T\mathcal{M}$ be a one parameter family of geodesics on $T\mathcal{M}$ where $\tau \in (-\epsilon, \epsilon)$, and $\bar{\gamma} := \bar{\gamma}_0$. A vector field $\bar{J}(t) \in T_{\bar{\gamma}(t)}T\mathcal{M}$ given by

$$\bar{J}(t) := \frac{\partial \bar{\gamma}_\tau}{\partial \tau}(t)\Big|_{\tau=0},$$

is said to be a *Jacobi field on* $T\mathcal{M}$ along the geodesic $\bar{\gamma}$. Consequently, a Jacobi field $\bar{J}$ along γ satisfies the so-called *Jacobi equation*

$$\overline{\nabla}_X \overline{\nabla}_X \bar{J} = \bar{R}(X, \bar{J})X.$$

Lemma

The differential of the geodesic flow map $\phi_t : T\mathcal{M} \to T\mathcal{M}$ *is given by*

$$\bar{J}(t) = \mathrm{Hor}_{\phi_t(x,p)}(J(t)) + \mathrm{Ver}_{\phi_t(x,p)}(\nabla_{\dot{\gamma}} J(t)),$$

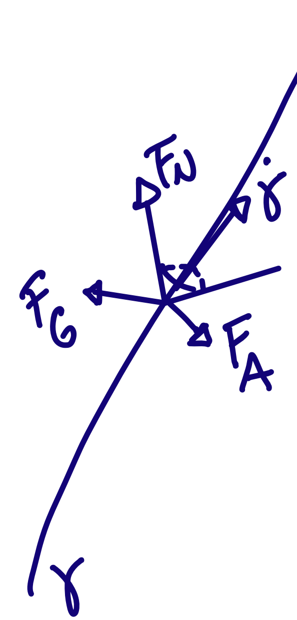
for every $t \in \mathbb{R}$*, every* $(x,p) \in T\mathcal{M}$*, and every vector* $V \in T_{(x,p)}T\mathcal{M}$*.*

Jacobi fields along null geodesics in spacetime I

Let us consider a double null frame along a null geodesic γ in $\mathcal{M}$ given by

$$F_N, \quad F_G, \quad F_A, \quad \dot{\gamma},$$

where F_G, F_A are spacelike vector fields, and $\dot{\gamma}$, F_N are null vector fields.



Jacobi fields along null geodesics in spacetime I

Let us consider a double null frame along a null geodesic γ in $\mathcal{M}$ given by

$$F_N, \quad F_G, \quad F_A, \quad \dot{\gamma},$$

where F_G, F_A are spacelike vector fields, and $\dot{\gamma}$, F_N are null vector fields.

The Jacobi equation along γ is given by an ode system where

$$\frac{d^2}{dt^2}\Big(J^G - \frac{2Es}{l}J^N\Big) = \Big(J^G - \frac{2Es}{l}J^N\Big)\Big(\frac{3mD}{r^3} - \frac{3m}{r^3D}\Big(\frac{Dp^r}{E}\Big)^2\Big)$$
$$+ \frac{d}{dt}\Big(J^G - \frac{2Es}{l}J^N\Big)\Big(\frac{2m}{Dr^2}\frac{Dp^r}{E}\Big) - \frac{4D^2}{lE}\frac{dJ^N}{ds} + \mathrm{Err},$$

with respect to the time coordinate t.

Jacobi fields in a neighborhood of the trapped set

Using the Riccati equation for $J^G - \frac{2Es}{l} J^N$ given by

$$\frac{d}{dt}\Big[\frac{d}{dt}\log\Big(J^G - \frac{2Es}{l}J^N\Big)\Big] + \Big[\frac{d}{dt}\log\Big(J^G - \frac{2Es}{l}J^N\Big)\Big]^2 = \frac{3m(r-2m)}{r^4} - \frac{3m}{r^3 D}\Big(\frac{Dp^r}{E}\Big)^2 + \frac{d}{dt}\log\Big(J^G - \frac{2Es}{l}J^N\Big)\frac{2m}{Dr^2}\frac{Dp^r}{E} - \frac{4D^2}{lE}\frac{dJ^N}{ds}\Big(J^G - \frac{2Es}{l}J^N\Big)^{-1} + \mathrm{Err}.$$

A standard argument using a family of closed invariant cones as in Katok–Hasselblatt's book, we obtain a solution $q_+ > 0$ to the Riccati equation.

Improving the decay of $\partial_r \mathrm{T}_{uv}$

Proposition

Under the bootstrap assumptions, the energy momentum tensor satisfies

$$|\partial_r \mathrm{T}_{uv}| \leq \frac{C\epsilon}{\exp((\frac{2}{3\sqrt{3}m} - \frac{\delta}{2})v)},$$

for every $(u, v) \in \mathcal{R}_{3m}$, *where* $C > 0$ *is a constant depending on* f_0, δ, R.

The radial derivative of the energy momentum tensor satisfies

$$\partial_r \mathrm{T}_{uv}(f) = \rho(\Omega^2 p^u \Omega^2 p^v \mathrm{Hor}_{(x,p)}(\partial_r)(f)) + \mathrm{Err}.$$

Decomposing the radial vector field

$$\mathrm{Hor}_{(x,p)}(\partial_r) = \frac{Er}{lD}\mathrm{Hor}_{(x,p)}(F_G) - \frac{r^2}{2l^2 D}\left(p^r - \frac{2E^2 s}{r}\right)\mathrm{Hor}_{(x,p)}(\dot{\gamma}),$$

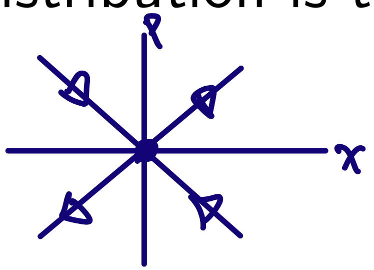
we can use the unstable vector field.

A Vlasov equation with a trapping potential

Let $f : [0, \infty) \times \mathbb{R}_x \times \mathbb{R}_p \to [0, \infty)$ be a distribution satisfying the Vlasov equation with the potential $\frac{-|x^2|}{2}$,

$$\partial_t f + p \cdot \partial_x f + x \cdot \partial_p f = 0. \tag{1}$$

The distribution is transported along the Hamiltonian flow



$$\dot{x} = p, \qquad \dot{p} = x.$$

Proposition

Let f_0 be a compactly supported data for (1). Then, the solution of the Vlasov equation satisfies

$$|\partial_x^n \rho(f)(t, x)| \leq \frac{L_n(f_0)}{\exp((n+1)t)} \tag{2}$$

*Weight.-Sobolev.

The massive Vlasov equation on Schwarzschild

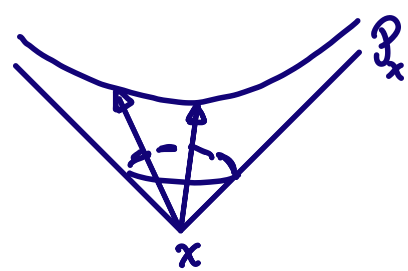
Let us investigate the linear dynamics of a distribution $f(x,p)$ satisfying the *massive Vlasov equation on Schwarzschild spacetime* (Schw, g_s) given by

$$Xf = 0,$$

in terms of the generator of the geodesic flow $X \in TT\mathrm{Schw}$.

The distribution function $f : \mathcal{P}_1 \to [0,\infty)$ is defined on the *mass-shell* $\mathcal{P}_1$, given by

$$\mathcal{P}_1 := \Big\{ (x,p) \in T\mathrm{Schw} : g_x(p,p) = -1, \text{ where } p \text{ is future directed} \Big\}.$$



Previous results – On massive Vlasov

1. Stability of Minkowski spacetime for the Einstein–massive Vlasov system (Lindblad–Taylor 2020, Fajman–Joudioux–Smulevici 2021, Wang 2022).

Previous results – On massive Vlasov

1. Stability of Minkowski spacetime for the Einstein–massive Vlasov system (Lindblad–Taylor 2020, Fajman–Joudioux–Smulevici 2021, Wang 2022).

2. Static solutions for the spherically symmetric Einstein–massive Vlasov system (Rein–Rendall 1993, Jabiri 2021). *Bifurcation

3. Results on phase mixing for the massive Vlasov equation on Schwarzschild (Rioseco–Sarbach 2018).

Setup of the main result I

The standard *particle energy* E, the *total angular momentum* l, and the *azimuthal angular momentum* l_ϕ, defined by

$$E := D(r)(p^u + p^v), \quad l := r^2\sqrt{(p^\theta)^2 + \sin^2\theta(p^\phi)^2}, \quad l_\phi := r^2 \sin^2\theta p^\phi,$$

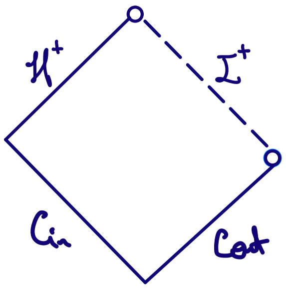
are conserved quantities along the geodesic flow.

Setup of the main result I

The standard *particle energy* E, the *total angular momentum* l, and the *azimuthal angular momentum* l_ϕ, defined by

$$E := D(r)(p^u + p^v), \quad l := r^2\sqrt{(p^\theta)^2 + \sin^2\theta(p^\phi)^2}, \quad l_\phi := r^2\sin^2\theta p^\phi,$$

are conserved quantities along the geodesic flow.
We define the invariant region $\mathcal{D}_0$ given by



$$\mathcal{D}_0 := \Big\{(x,p) \in \mathcal{P} : l > 4M, \quad E > 1\Big\},$$

where almost every timelike geodesic either crosses the event horizon or is unbounded. Let us define the subset Σ_0 given by

$$\Sigma_0 = \Big\{(x,p) \in \mathcal{P} : x \in \underline{C}_{in} \cup C_{out}, \quad l > 4M, \quad E > 1\Big\},$$

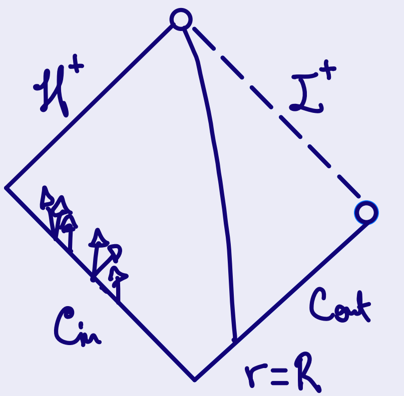
where we will assume the initial distribution function is supported.

The main result I

Theorem (V.R. 2022)

Let f_0 be an initial data for (V) that is compactly supported on Σ_0. Then, there exists $R > 2M$, such that the energy momentum tensor for the solution f to (V) satisfies

$$\mathrm{T}_{uv} \leq \frac{C_0}{u^3},$$



for all $x \in \{r \geq R\}$; and

$$\mathrm{T}_{uv} \leq \frac{C_1(1-\frac{2M}{r})}{\exp(\frac{1}{4\sqrt{2}M}v)},$$

for all $x \in \{r \leq R\}$, where C_0, and C_1 are positive constants depending on f_0, R, and M.

Setup of the main result II

Let $l \in [2\sqrt{3}M, \infty)$, there exist geodesics with angular momentum l that are contained in the spheres $\{r = r_\pm(l)\}$ of radii $r_-(l)$ and $r_+(l)$, that are determined by

$$r^2 - \frac{l^2}{M} r + 3l^2 = 0,$$

where $r_-(l) \leq r_+(l)$.

Setup of the main result II

Let $l \in [2\sqrt{3}M, \infty)$, there exist geodesics with angular momentum l that are contained in the spheres $\{r = r_{\pm}(l)\}$ of radii $r_{-}(l)$ and $r_{+}(l)$, that are determined by

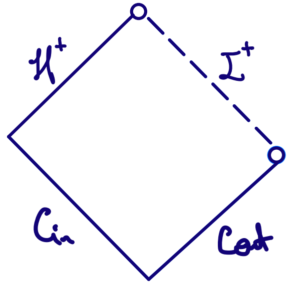
$$r^2 - \frac{l^2}{M} r + 3l^2 = 0,$$

where $r_{-}(l) \leq r_{+}(l)$.We define the invariant region $\mathcal{D}$ given by

$$\mathcal{D} = \Big\{ (x,p) \in \mathcal{P} : \; l \geq 4M \text{ such that if } E < 1 \text{ then } r < r_{-}(l) \Big\}$$
$$\cup \Big\{ (x,p) \in \mathcal{P} : \; l < 4M \text{ such that if } E < E_{-}(l) \text{ then } r < r_{-}(l) \Big\},$$

where almost every geodesic either crosses the event horizon or is unbounded. We define the set Σ given by

$$\Sigma = \Big\{ (x,p) \in \mathcal{D} : x \in \underline{C}_{in} \cup C_{out} \Big\}$$



where we will assume the initial distribution function is supported.

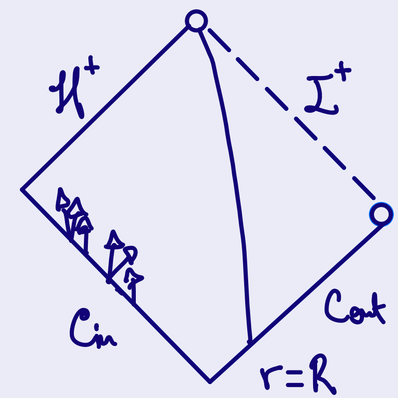
The main result II

Theorem (V.R. 2022)

Let f_0 be an initial data for (V) that is compactly supported on Σ. Then, there exists $R > 2M$, such that the energy momentum tensor for the solution f to (V) satisfies

$$\mathrm{T}_{uv} \leq \frac{C_0}{u^{\frac{1}{3}} r^2},$$

for all $x \in \{r \geq R\}$; and



$$\mathrm{T}_{uv} \leq \frac{C_1}{v^{\frac{1}{3}}}\Big(1 - \frac{2M}{r}\Big),$$

for all $x \in \{r \leq R\}$, where C_0, and C_1 are positive constants depending on f_0, R, and M.

The timelike geodesic flow in Schwarzschild I

The geodesic equation for the radial coordinate is given by

$$\frac{dr}{ds} = p^r, \qquad \frac{dp^r}{ds} = -\frac{M}{r^4}\Big(r^2 - \frac{l^2}{M}r + 3l^2\Big),$$

where the particle energy can be written as

$$E^2 = (p^r)^2 + \Big(1 - \frac{2M}{r}\Big)\Big(1 + \frac{l^2}{r^2}\Big), \qquad V_l(r) := \Big(1 - \frac{2M}{r}\Big)\Big(1 + \frac{l^2}{r^2}\Big).$$

The timelike geodesic flow in Schwarzschild I

The geodesic equation for the radial coordinate is given by

$$\frac{dr}{ds} = p^r, \qquad \frac{dp^r}{ds} = -\frac{M}{r^4}\Big(r^2 - \frac{l^2}{M}r + 3l^2\Big),$$

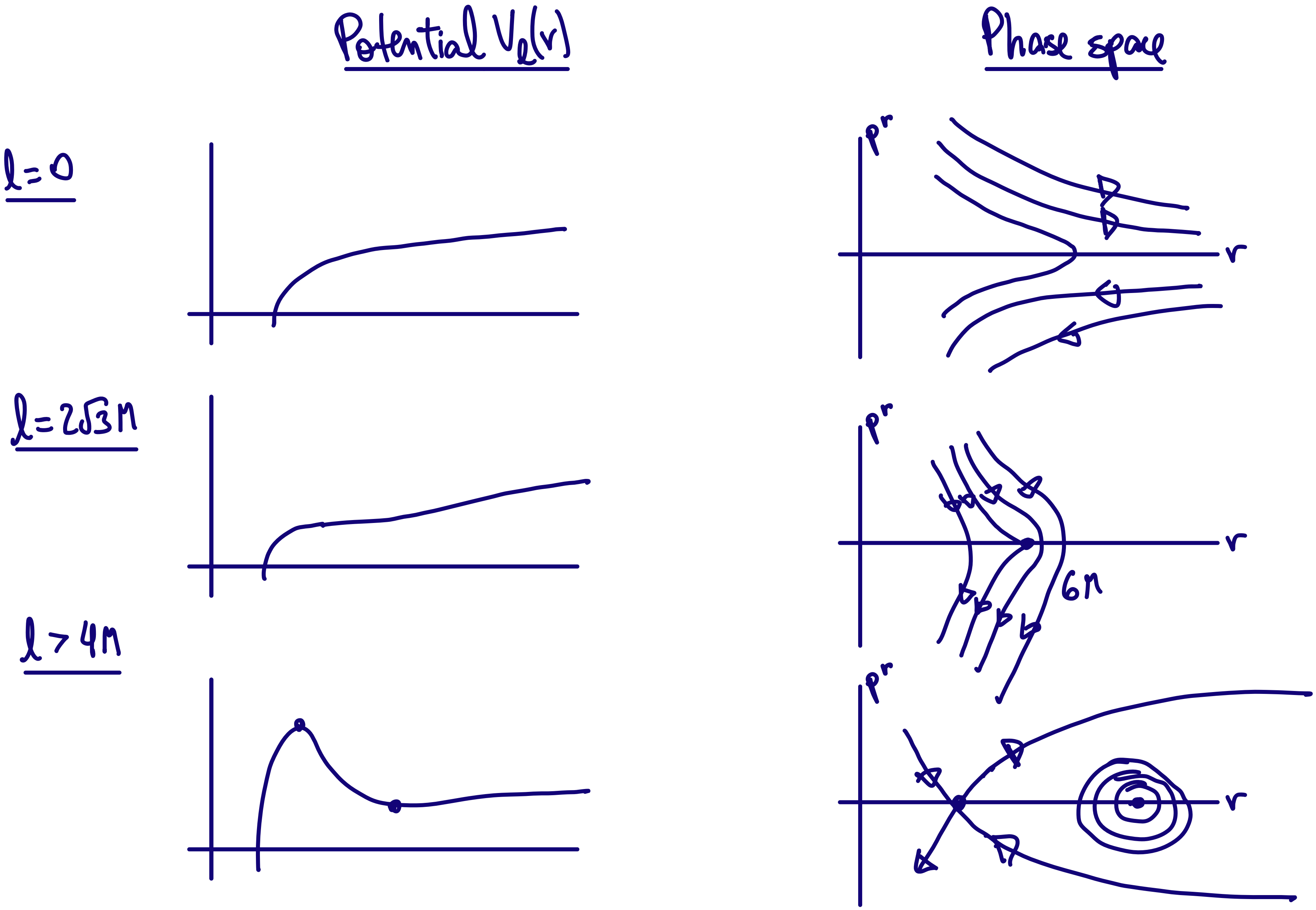
where the particle energy can be written as

$$E^2 = (p^r)^2 + \Big(1 - \frac{2M}{r}\Big)\Big(1 + \frac{l^2}{r^2}\Big), \qquad V_l(r) := \Big(1 - \frac{2M}{r}\Big)\Big(1 + \frac{l^2}{r^2}\Big).$$

The radial geodesic equation for the radial coordinate with respect to the time coordinate t is given by

$$\frac{d^2r}{dt^2} = -\frac{MD^2}{r^4V_l}\Big(r^2 - \frac{l^2}{M}r + 3l^2\Big) + \frac{3M}{r^4V_l}\Big(r^2 - \frac{l^2}{3M}r + \frac{5l^2}{3}\Big)\Big(\frac{p^r}{p^t}\Big)^2.$$

The timelike geodesic flow in Schwarzschild II



Key lemmata for the decay of T_{uv}

Lemma

Let $R > r_s > 2M$. There exists $C > 0$ such that for every geodesic γ contained in $\{r > r_s\}$ with $l \leq 2\sqrt{3}M$ and $E \in [\frac{95}{100}, 1)$, we have

$$\frac{2M}{r} - \frac{l^2}{r^2} + \frac{2Ml^2}{r^3} - (p^r)^2 \leq \frac{C}{v^{\frac{2}{3}}(s)},$$

when the radial coordinate satisfies $r(0) < R$.

This lemma is obtained by integrating in time the radial geodesic equation in a neighborhood of $\{(x, p) \in \mathcal{P} : E = 1\}$.

Key lemmata for the decay of T_{uv}

Lemma

Let $R > r_s > 2M$. There exists $C > 0$ such that for every geodesic γ contained in $\{r > r_s\}$ with $l \leq 2\sqrt{3}M$ and $E \in [\frac{95}{100}, 1)$, we have

$$\frac{2M}{r} - \frac{l^2}{r^2} + \frac{2Ml^2}{r^3} - (p^r)^2 \leq \frac{C}{v^{\frac{2}{3}}(s)},$$

when the radial coordinate satisfies $r(0) < R$.

This lemma is obtained by integrating in time the radial geodesic equation in a neighborhood of $\{(x, p) \in \mathcal{P} : E = 1\}$.

Thank you for your attention!