Soliton resolution for energy-critical wave maps in the equivariant case

Jacek Jendrej
(CNRS & Université Sorbonne Paris Nord)
joint work with Andrew Lawrie
(Massachusetts Institute of Technology)

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Energy-critical wave maps equation

"Definition" An application $\Psi : \mathbb{R}^{1+2} \to S^2 \subset \mathbb{R}^3$ is a wave map if it is a critical point of the Lagrangian

$$\mathcal{L}(\Psi, \partial_t \Psi, \nabla_x \Psi) := \frac{1}{2} \iint \left(|\partial_t \Psi|^2 - |\nabla_x \Psi|^2 \right) dx\, dt \,.$$

Natural analogs of linear waves in a nonlinear, geometric setting.

Euler-Lagrange equation:

$$\partial_t^2 \Psi(t,x) - \Delta_x \Psi(t,x) = -\left(|\partial_t \Psi(t,x)|^2 - |\nabla_x \Psi(t,x)|^2 \right) \Psi(t,x).$$

Local well-posedness and global well-posedness for small data:

Klainerman, Selberg, Machedon, Tataru, Tao, ... (1993- 2002)

$\boxed{\text{Equivariant wave maps}}$

We study the dynamics (long time behavior) of large solutions, but only in a special case:

$$\Psi(t, r\cos\theta, r\sin\theta) = (\sin\psi(t,r)\cos(k\theta), \sin\psi(t,r)\sin(k\theta), \cos\psi(t,r)).$$

Here, $k \in \{1, 2, \ldots\}$ is the equivariance degree, $t \in \mathbb{R}$, $r \in (0, \infty)$

Equation for ψ:

$$\boxed{(WM_k) \quad \partial_t^2 \psi(t,r) - \partial_r^2 \psi(t,r) - \frac{1}{r}\partial_r \psi(t,r) + \frac{k^2}{2r^2}\sin(2\psi(t,r)) = 0.}$$

Lagrangian: $\mathcal{L} := \pi \iint \left((\partial_t \psi)^2 - (\partial_r \psi)^2 - \frac{k^2 \sin^2 \psi}{r^2} \right) r\, dr\, dt$

Energy: $E(\psi_0, \dot\psi_0) := \pi \int_0^\infty \Big(\underbrace{\dot\psi_0^{\,2}}_{\text{kinetic}} + \underbrace{(\partial_r \psi_0)^2 + \frac{k^2 \sin^2 \psi_0}{r^2}}_{\text{potential}} \Big) r\, dr$

Scaling invariance and criticality

If ψ solves (WM_k) and $\lambda > 0$, then

$$\psi_\lambda(t,r) := \psi\left(\tfrac{t}{\lambda}, \tfrac{r}{\lambda}\right) \quad \text{solves } (WM_k) \text{ as well.}$$

Moreover, $E(\psi_\lambda, \dot\psi_\lambda) = E(\psi, \dot\psi)$ $\rightsquigarrow$ energy-critical problem

Note: if $\lambda \ll 1$, then ψ_λ is concentrated and evolves fast.

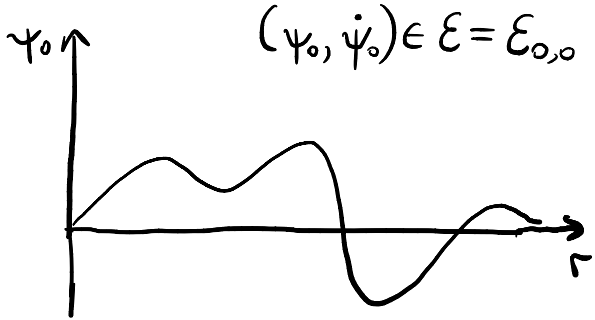
Local theory, small data theory

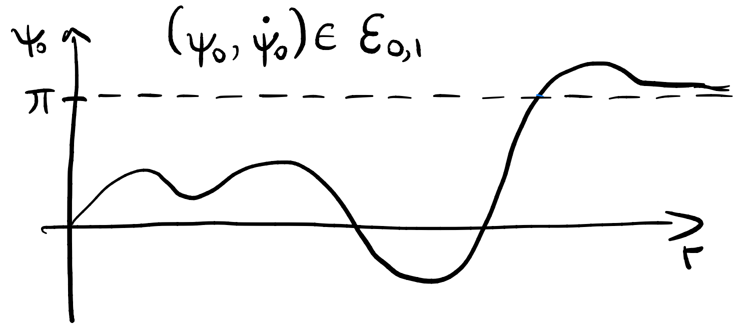
Energy norm: $\|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 := \|\dot\psi_0\|_{L^2}^2 + \|\psi_0\|_{\mathcal{H}}^2$,

$$\|\dot\psi_0\|_{L^2}^2 := \int_0^\infty \dot\psi_0^2 \, r\,dr, \qquad \|\psi_0\|_{\mathcal{H}}^2 := \int_0^\infty \left((\partial_r \psi_0)^2 + \frac{k^2}{r^2}\psi_0^2 \right) r\,dr.$$

$\|\psi_0\|_{L^\infty}$ small $\Rightarrow$ $\|(\psi_0, \dot\psi_0)\|_{\mathcal{E}}^2 \simeq E(\psi_0, \dot\psi_0)$.

Finite energy sectors: $\mathcal{E}_{m,n} := \left\{ (\psi_0, \dot\psi_0) : E(\psi_0, \dot\psi_0) < \infty, \ \lim_{r\to 0} \psi_0(r) = m\pi, \ \lim_{r\to\infty} \psi_0(r) = n\pi \right\}$.





Theorem (Shatah–Struwe 1994, $k=1$
Burq, Planchon–Stalker–Tahvildar Zadeh 2003, $k \geq 2$)

Equation (WM_k) is locally well-posed in each finite energy sector, in the sense of strong solutions.

Linearisation around $\psi = 0$: $\partial_t^2 \psi_L - \partial_r^2 \psi_L - \frac{1}{r}\partial_r \psi_L + \frac{k^2}{r^2}\psi_L = 0$.

If ψ is small, the nonlinear effects become negligible for large times:

Theorem If $E(\psi_0, \dot\psi_0)$ is small enough, then ψ exists globally and

$$\lim_{t \to \pm\infty} \left\| \left(\psi(t), \partial_t \psi(t)\right) - \left(m\pi + \psi_L^{\pm}(t), \partial_t \psi_L^{\pm}(t)\right) \right\|_{\mathcal{E}} = 0 \qquad \text{(scattering)}$$

These results are consequences of Strichartz estimates.

$\boxed{\text{Stationary solutions ("solitons" or "bubbles")}}$

Minimisers of E:

* on $\mathcal{E}_{m,m}$ $\rightsquigarrow$ constant functions
* on $\mathcal{E}_{m,m+1}$ $\rightsquigarrow$ $(m\pi + 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$
* on $\mathcal{E}_{m,m-1}$ $\rightsquigarrow$ $(m\pi - 2\arctan(r^k/\lambda^k), 0)$, $\lambda > 0$
* on other sectors $\rightsquigarrow$ $\emptyset$

We denote $Q(r) := 2\arctan(r^k)$, $Q_\lambda := Q(r/\lambda)$ for $\lambda > 0$.

- Key role in the description of the dynamics of large solutions.

Absence of self-similarity

Theorem (Christodoulou, Tahvildar-Zadeh, Shatah 1992)

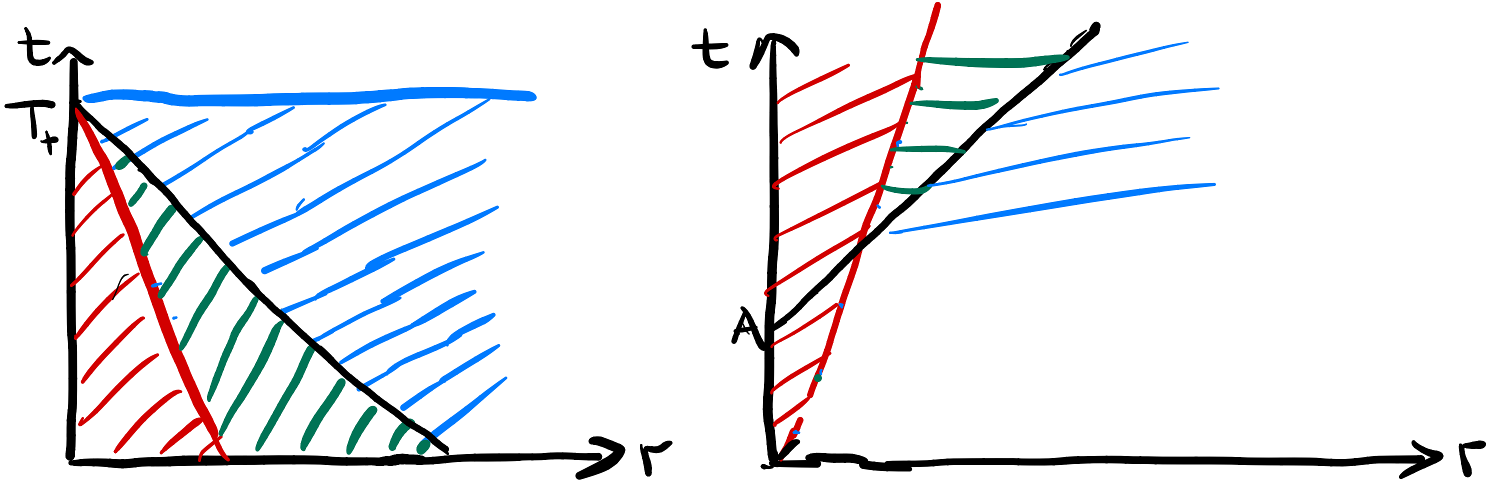
If $T_+ < \infty$ the max. time of existence of ψ, then for every $\alpha > 0$

$$\lim_{t \to T} \int_{\alpha(T_+ - t)}^{T_+ - t} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{k^2}{r^2} \psi^2 \right] r \, dr = 0.$$

Theorem (Côte, Kenig, Lawrie, Schlag 2015)

If $T_+ = \infty$ the max. time of existence of ψ, then for every $\alpha > 0$

$$\lim_{A \to \infty} \limsup_{t \to \infty} \int_{\alpha t}^{t - A} \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{k^2}{r^2} \psi^2 \right] r \, dr = 0.$$



(The energy in the green region $\longrightarrow 0$ as $t \to T_+$.)

<u>**Theorem**</u> (Struwe 2003)

If $T_+ < \infty$, then there exist $t_n \to T_+$, $m \in \mathbb{Z}$, $\iota \in \{-1, 1\}$ and $0 < \lambda_n \ll T_+ - t_n$ such that for all $R > 0$

$$\lim_{n\to\infty} \left\| \left(\psi(t_n, \lambda_n \cdot), \lambda_n \partial_t \psi(t_n, \lambda_n \cdot)\right) - (m\pi + \iota Q, 0) \right\|_{\mathcal{E}(r \le R)} = 0.$$

Sequential soliton resolution

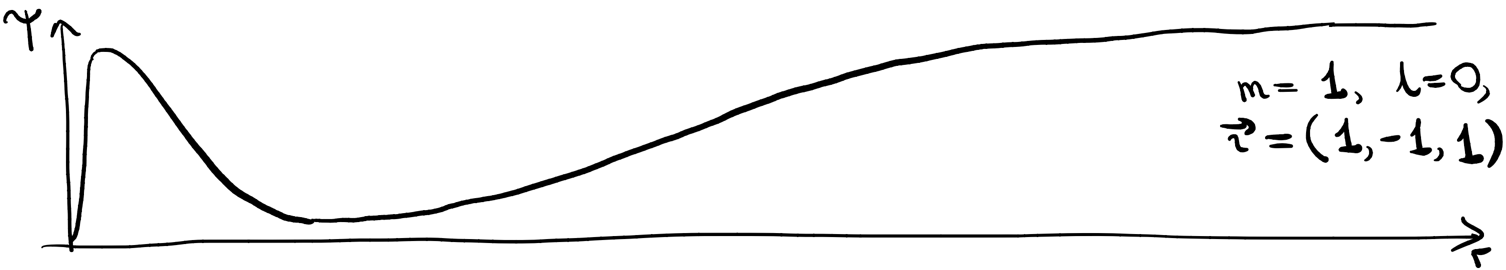
Given $M \in \mathbb{N}$, $m \in \mathbb{Z}$, $\vec{\iota} = (\iota_1, \dots, \iota_M) \in \{-1, 1\}^M$,
$\vec{\lambda} = (\lambda_1 \dots \lambda_M) \in (0, \infty)^M$, $\lambda_1 < \lambda_2 < \dots < \lambda_M$,

we denote

$$Q(m, \vec{\iota}, \vec{\lambda}; r) := m\pi + \sum_{j=1}^{M} \iota_j \left(Q_{\lambda_j} - \pi \right) \in \mathcal{E}_{\ell, m}$$

with $\ell = m - \sum_{j=1}^{M} \iota_j$.

If $\lambda_1 \ll \lambda_2 \ll \dots \ll \lambda_M$, we call $Q(m, \vec{\iota}, \vec{\lambda})$ a <u>multi-bubble configuration</u>.



Sequential soliton resolution

Theorem (Côte 2015, Jia–Kenig 2017)

① If $T_+ < \infty$, then there exist $(\psi_*, \dot\psi_*)$, M, m, $\vec{\iota}$, $\vec{\lambda}_n$, $t_n \to T_+$:

$$\lim_{n\to\infty} \Big[\|\psi(t_n) - \psi_* - Q(m, \vec{\iota}, \vec{\lambda}_n)\|_{\mathcal{H}}^2 + \|\partial_t \psi(t_n) - \dot\psi_*\|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_{n,j}/\lambda_{n,j+1}\right)^k + \left(\lambda_{n,M}/(T_+ - t_n)\right)^k \Big] = 0.$$

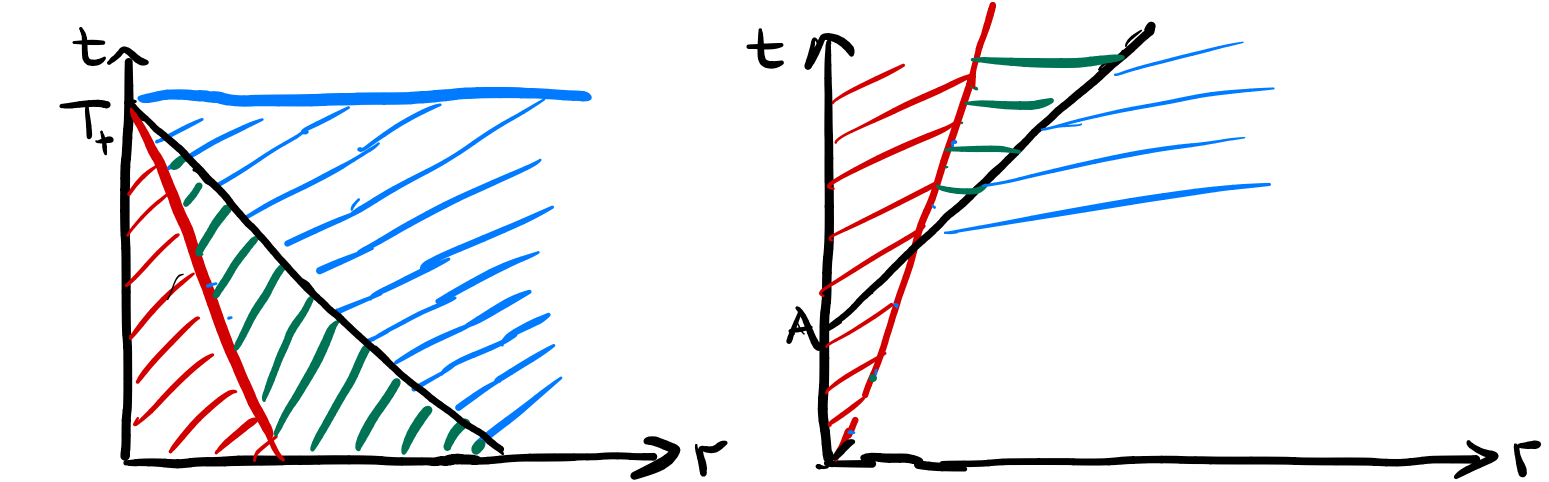
② If $T_+ = \infty$, then there exist $(\psi_L, \partial_t \psi_L)$, M, m, $\vec{\iota}$, $\vec{\lambda}_n$, $t_n \to \infty$:

$$\lim_{n\to\infty} \Big[\|\psi(t_n) - \psi_L(t_n) - Q(m, \vec{\iota}, \vec{\lambda}_n)\|_{\mathcal{H}}^2 + \|\partial_t \psi(t_n) - \partial_t \psi_L(t_n)\|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_{j,n}/\lambda_{j+1,n}\right)^k + \left(\lambda_{M,n}/t_n\right)^k \Big] = 0.$$

Remark Convergence in continuous time outside of cones:

① $\forall \alpha > 0$: $\lim_{t\to T_+} \left(\|\psi(t) - \psi_*\|^2_{\mathcal{H}(r \geq \alpha(T_+ - t))} + \|\partial_t \psi(t) - \dot\psi_*\|^2_{L^2(r \geq \alpha(T_+ - t))} \right) = 0.$

② $\forall \alpha > 0$: $\lim_{t\to\infty} \left(\|\psi(t) - \psi_L(t)\|^2_{\mathcal{H}(r \geq \alpha t)} + \|\partial_t \psi(t) - \partial_t \psi_L(t)\|^2_{L^2(r \geq \alpha t)} \right) = 0.$



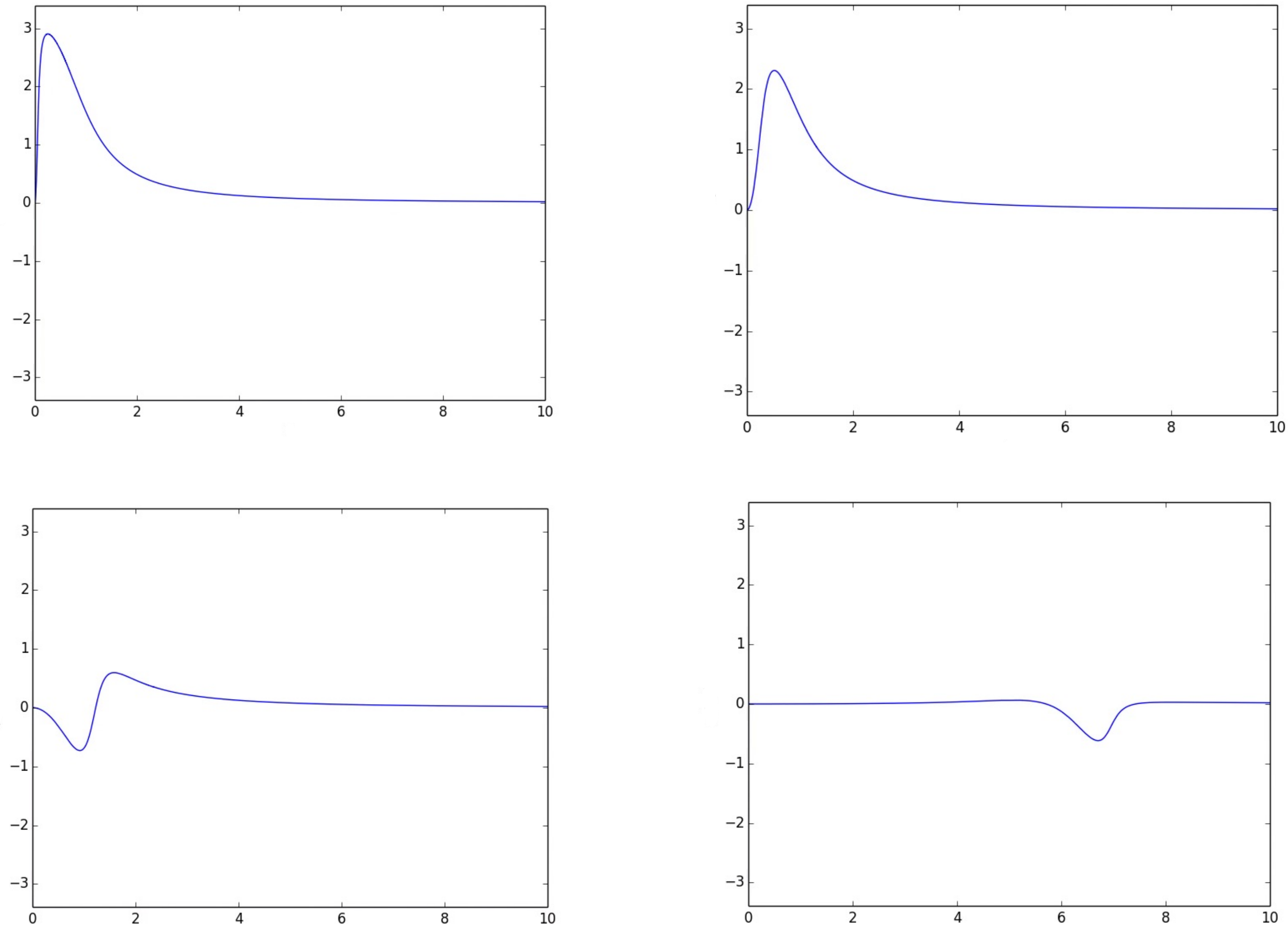
$E \longrightarrow ME(Q)$

$E \longrightarrow 0$

$E \longrightarrow E(\psi_*, \dot\psi_*)$ (resp. $E_L(\psi_L, \partial_t \psi_L)$)

Question : Does the decomposition hold in continuous time?

Ennemy : Collisions in the red region.
(Excluded if all bubbles have the same sign.)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Continuous time soliton resolution

Theorem (J.–Lawrie 2021)

① If $T_+ < \infty$, then there exists $\vec{\lambda} : [0, T_+) \to (0,\infty)^M$ such that

$$\lim_{t \to T_+} \Big[\| \psi(t) - \psi_* - Q(m, \vec{\iota}, \vec{\lambda}(t)) \|_H^2 + \| \partial_t \psi(t) - \dot{\psi}_* \|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_j(t) / \lambda_{j+1}(t) \right)^k + \left(\lambda_M(t) / (T_+ - t) \right)^k \Big] = 0.$$

② If $T_+ = \infty$, then there exists $\vec{\lambda} : [0, \infty) \to (0,\infty)^M$ such that

$$\lim_{t \to \infty} \Big[\| \psi(t) - \psi_L(t) - Q(m, \vec{\iota}, \vec{\lambda}(t)) \|_H^2 + \| \partial_t \psi(t) - \partial_t \psi_L(t) \|_{L^2}^2 + \sum_{j=1}^{M-1} \left(\lambda_j(t) / \lambda_{j+1}(t) \right)^k + \left(\lambda_M(t) / t \right)^k \Big] = 0.$$

Some related results

* Duyckaerts, Kenig, Merle 2010 — sequential soliton resolution for $\partial_t^2 u - \Delta_x u - u^5 = 0$ in 3D.
* Duyckaerts, Kenig, Merle 2012 — continuous time; energy channels
* J.–Lawrie 2017 — classification for $E \leqslant 2E(Q)$, $k \geqslant 2$; crucially used estimates of interactions between solitons (reduction to a "2-body problem")
* Rodriguez 2019 ——— " ——— $k=1$
* D, K, M 2019 : generalisation of 2012 result to all odd dim.
* Duyckaerts, Kenig, Martel, Merle 2021 — settled $k=1$ using energy channels

Existence of bubbles and multi-bubbles

The answer depends on the equivariance class:

* $k = 1, 2, 3, \ldots$: blow-up solutions with $M = 1$ bubble
  (Krieger – Schlag – Tataru '08, Rodnianski – Sterbenz 2010
  Raphaël – Rodnianski 2011)

  blow-up with $M \geq 2$ unknown

* many other constructions for $M = 1$ exist.

* $k = 2, 3, \ldots$: existence, uniqueness and asymptotic description
  of pure two-bubble solutions (J '16, J – Lawrie 17–20)

* $k = 1$: no solutions with $M \geq 2$ are known;
  no pure two-bubbles exist (Rodriguez '18)
  no pure multi-bubbles exist (J. – Lawrie, unpublished)

Threshold behavior : the two-bubbles

Simplest case not settled by the sequential result :

$$M = 2, \qquad \iota_1 = 1, \qquad \iota_2 = -1.$$

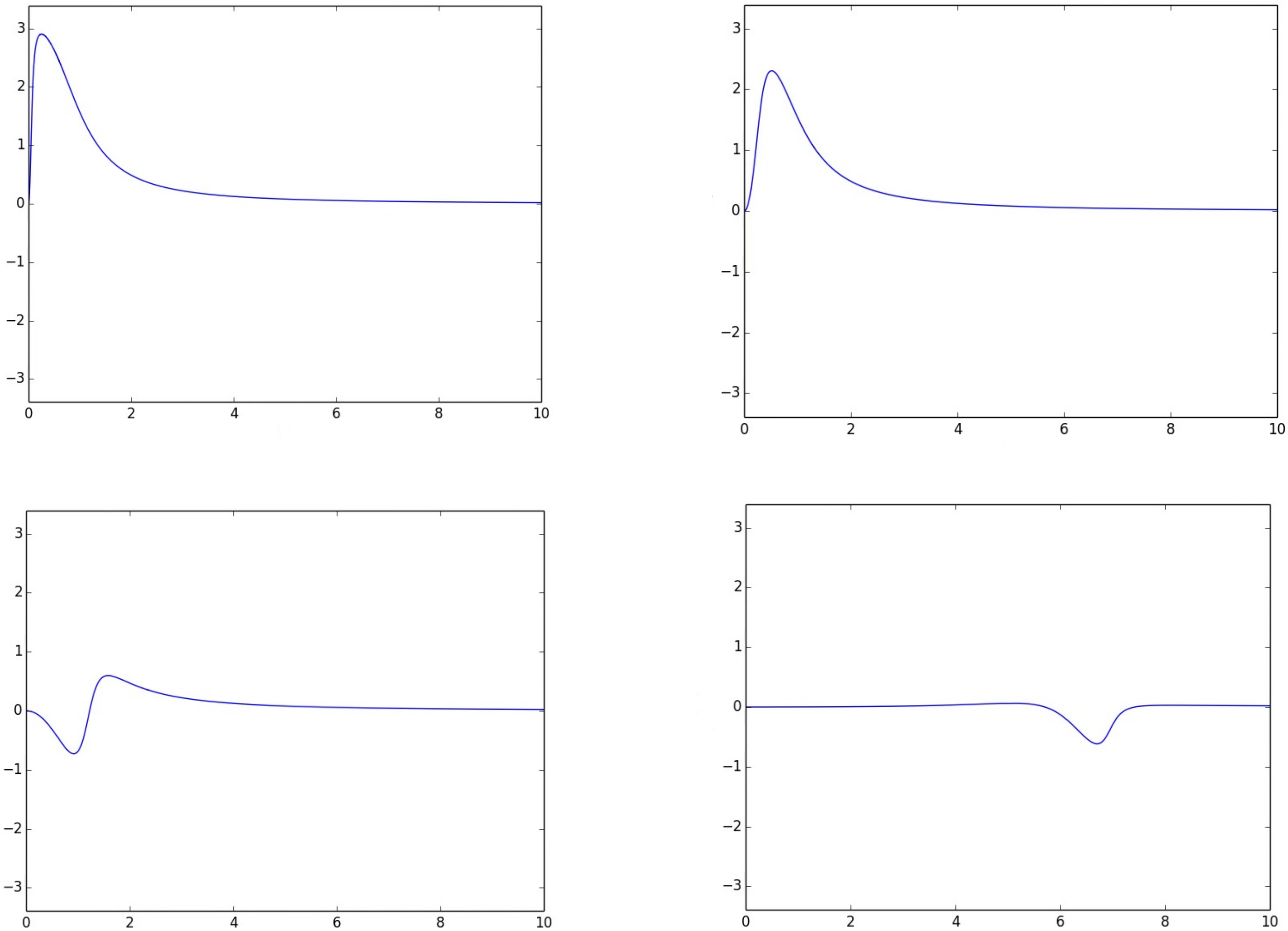
Minimal possible energy : $2E(Q) = 8k\pi$.

<u>Theorem</u> (J.–PhD thesis 2016)

Let $k \geqslant 2$. There exists a solution $(\psi_c, \dot\psi_c)$ of (WM_k) such that

$$\lim_{t \to \infty} \left\| (\psi_c, \dot\psi_c) - \left(Q_{\kappa t^{-\frac{2}{k-2}}} - Q, 0\right) \right\|_{\mathcal{E}} = 0,$$

where κ is an explicit constant depending on k.



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Theorem (J.–Lawrie 2018)

Let $k \geq 2$. If $(\psi, \dot\psi)$ is a solution of (WM_k) in the sector $\mathcal{E} = \mathcal{E}_{0,0}$ of energy $E(\psi, \dot\psi) \leq 8k\pi$,

then either $(\psi, \dot\psi)$ scatters in both time directions, or forms a two-bubble in one time direction (in cont. time) and scatters in the other time direction.

Theorem (J.–Lawrie 2020)

Let $k \geq 4$. If $(\psi, \dot\psi)$ is a solution of (WM_k) in the sector $\mathcal{E} = \mathcal{E}_{0,0}$ of energy $E(\psi, \dot\psi) \leq 8k\pi$,

then either $(\psi, \dot\psi)$ scatters in both time directions,

or there exist $\iota, \sigma \in \{-1, 1\}$, $\lambda > 0$ and $t_0 \in \mathbb{R}$ such that

$$\psi(t, r) = \iota\, \psi_c\left(\sigma \frac{t - t_0}{\lambda}, \frac{r}{\lambda}\right), \qquad \forall (t, r) \in \mathbb{R} \times (0, \infty)$$

Main ideas of the proof of soliton resolution

The desired decomposition holds for a time sequence.

We need to prevent the following scenario:

close to multi-bubble configuration $\rightsquigarrow$ "collision" $\rightsquigarrow$ far from multi-bubble configuration $\rightsquigarrow$ close to multi-bubble configuration

"collision (time) interval"

occuring an infinite number of times (a <u>no-return lemma</u>).

* Inspired by Duyckaerts–Merle, Nakanishi–Schlag, Krieger–Nakanishi–Schlag for <u>single</u> soliton which is linearly unstable.

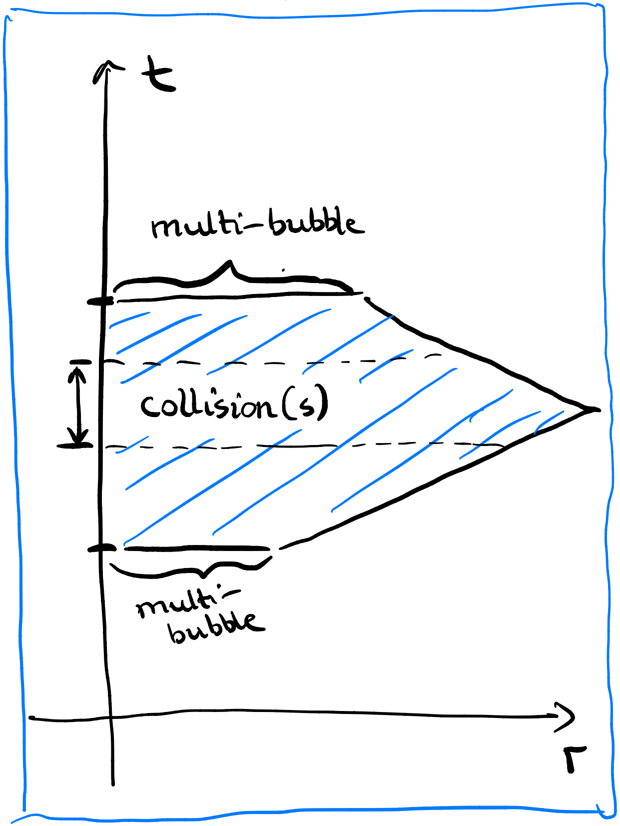
* Here, inter-soliton interactions play a similar role as linear instability in those works (cf. J.–Lawrie '17)

Virial identity

If ψ is a smooth wave map, then

$$\operatorname{div}_{t,r}\left(\partial_t\psi\, r^2\partial_r\psi,\ -\tfrac{1}{2}r^2(\partial_t\psi)^2-\tfrac{1}{2}r^2(\partial_r\psi)^2+\frac{k^2}{2}\sin^2\psi\right)=-r(\partial_t\psi)^2$$

We estimate the boundary terms from above and the space-time integral $\iint(\partial_t\psi)^2\, r\,dr\,dt$ from below



* collision duration $\gtrsim$ spatial scale
* "horizontal" boundary $\ll$ spatial scale
* $\iint(\partial_t\psi)^2\, r\,dr\,dt \gtrsim$ collision duration

("Compactness Lemma")

* remaining boundary : reduction to n-body

Interior and exterior bubbles

Let $K \in \{1, 2, \dots, N\}$, $[a,b] \subset [T_0, T_+)$, $0 < \varepsilon < \eta$.

We say $[a,b]$ is a collision interval with $N-K$ exterior bubbles if:

* $d(a) \leq \varepsilon$, $d(b) \leq \varepsilon$, $\exists\, c \in [a,b] : d(c) \geq \eta$

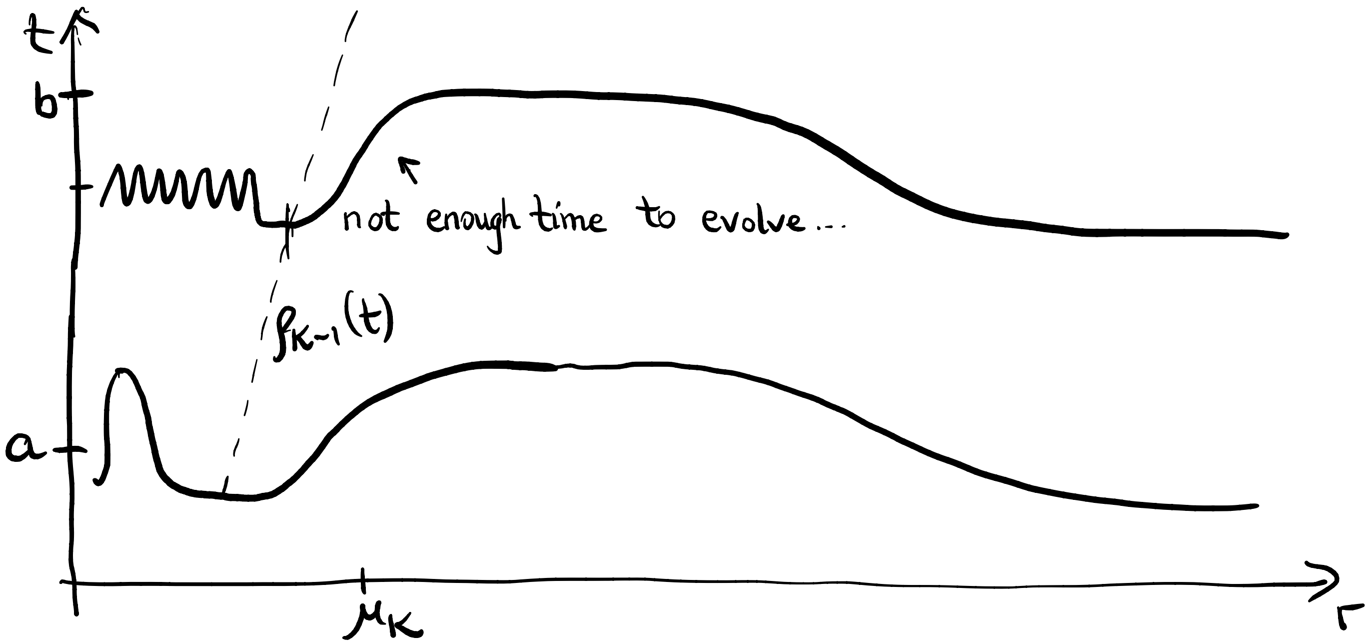
  where $d(t)$ is the distance of $\psi(t)$ to $\left(\begin{array}{c}\text{multi-bubble} \\ + \text{radiation}\end{array}\right)$

* $\exists$ curve $r = \rho_K(t)$ such that in the region $r \geq \rho_K(t)$ $\psi(t,r)$ is ε-close to $[(N-K)\text{-bubble} + \text{radiation}]$.

We now set K to be the <u>smallest</u> number such that there exist $\eta > 0$, a sequence $\varepsilon_n \to 0$ and a sequence of collision intervals $[a_n, b_n]$ corresponding to these parameters ε_n, η and K.

Lemma $\exists\, C = C(\psi) > 0$ and $\varepsilon > 0$ such that if $[a,b]$ collision interval with parameters (ε, η, K), then $b - a \geq C \min(\mu_K(a), \mu_K(b))$, where μ_K is the scale of the K-th bubble.

Proof If not, K would not be smallest possible.



Wave maps with small kinetic energy

<u>Lemma</u> ("Compactness Lemma") Let $\rho_n > 0$, $R_n \to \infty$, ψ_n defined for $t \in [0, \rho_n]$ of bounded energy such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} \int_0^{\rho_n} \int_0^{R_n \rho_n} (\partial_t \psi_n)^2 \, r \, dr \, dt = 0 .$$

Then, up to extraction of a subsequence, there exist $r_n \to \infty$, $t_n \in [0, \rho_n]$, $M, m, \vec{\iota}, \vec{\lambda}_n$ such that

$$\lim_{n\to\infty} \Big[\| \psi_n(t_n) - Q(m, \vec{\iota}, \vec{\lambda}_n) \|^2_{H(r \leq r_n \rho_n)} + \| \partial_t \psi_n(t_n) \|^2_{L^2(r \leq r_n \rho_n)} + \sum_{j=1}^{M-1} \left(\lambda_{n,j} / \lambda_{n,j+1} \right)^k + \left(\lambda_{n,M} / r_n \right)^k \Big] = 0$$

Modulation

* Near a_n and b_n, ψ is close to a multi-bubble and the analysis above does not apply.
* In this case, the main dynamical information are the scales of the bubbles
* We obtain differential inequalities on these scales.

Informally: $$\lambda_j'' \simeq -\iota_j \iota_{j+1} \omega^2 \frac{\lambda_j^{k-1}}{\lambda_{j+1}^k} + \iota_j \iota_{j-1} \omega^2 \frac{\lambda_{j-1}^k}{\lambda_j^{k+1}}.$$

* Error bounded by the energy of attractive interactions.
* The influence of the exterior bubbles and radiation can essentially be neglected by enlarging ε_n.
* Refined modulation parameters: Raphaël-Szeftel '11, J.-Lawrie '17.

Lemma If d starts growing at t_0, then $\forall\, t^* \geq t_0$

$$\int_{t_0}^{t_*} d(t)\,dt \leq C_0\, d(t_*)^{2/k} \mu_k(t_0) \quad \text{if } k \geq 2$$

$$\int_{t_0}^{t_*} d(t)\,dt \leq C_0\, d(t_*)^{2} \sqrt{-\log d(t_*)}\, \mu_k(t_0) \quad \text{if } k = 1.$$

The final step is to partition $[a_n, b_n]$ into

boundary term $\ll \mu_K$	$d \geq \theta_n \to 0$ boundary term absorbed by $\iint (\partial_t \psi)^2 r\,dr$	$d \geq \varepsilon$ $\iint (\partial_t \psi)^2 r\,dr \gtrsim \mu_K$	...

In the situation below, $\mu_K \simeq \text{const}$: