Global stability of Minkowski spacetime with minimal decay

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Evolution problem for Einstein vacuum equations

Einstein vacuum equations (EVE): $\mathbf{Ric}_{\mu\nu}(\mathcal{M}, \mathbf{g}) = 0$

Minkowski: $\mathcal{M} = \mathbb{R}^{1+3}$, $\mathbf{g} = -(dt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

Initial data of Minkowski: $\Sigma_0 = \mathbb{R}^3$, $g = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

Cauchy problem. For sufficiently smooth initial data, EVE are locally well-posed and there exists a unique maximal solution [Choquet-Bruhat 52'], [Choquet-Bruhat-Geroch 69']

Stability of Minkowski. If the spacetime is initially close to Minkowski, does it stay close and converge to Minkowski?

Previous works of Minkowski stability

[Christodoulou-Klainerman 93']. The global stability of Minkowski holds true for an asymptotically flat initial data

A large literature devoted to the stability of Minkowski for EVE: [Klainerman-Nicolò 03'], [Bieri 07'], [Lindblad-Rodnianski 10'], [Huneau 14'], [Hintz-Vasy 17'], [Graf 20'], [S. 22'], [Hintz 23'], [Huneau-Stingo-Wyatt 23']...

The stability of Minkowski coupled with matter fields: [Zipser 00'], [Loizelet 09'], [Speck 10'], [Lefloch-Ma 15', 22'], [Taylor 16'], [Wang, 16'], [Fajman-Joudioux-Smulevici 17'], [Lindblad-Taylor 17'], [Ionescu-Pausader 19'], [Bigorgne-Fajman-Joudioux-Smulevici-Thaller 20'], [Wang 22']...

(s,q)–asymptotically flat initial data

Definition. Given $s > 1$ and $q \in \mathbb{N}$, an initial data set (Σ_0, g, k) is called (s,q)–asymptotically flat if there exists a coordinate system (x^1, x^2, x^3) defined outside a sufficiently large compact set such that:

- In the case $s \geqslant 3$

$$g_{ij} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\sigma_{\mathbb{S}^2} + o_{q+1}(r^{-\frac{s-1}{2}}), \quad k_{ij} = o_q(r^{-\frac{s+1}{2}})$$

- In the case $1 < s < 3$

$$g_{ij} = \delta_{ij} + o_{q+1}(r^{-\frac{s-1}{2}}), \qquad k_{ij} = o_q(r^{-\frac{s+1}{2}})$$

Remark. The parameter s denotes the flux weight of curvature R, i.e. $\int_{\Sigma_0} r^s |R|^2 < \infty$ and q denotes the regularity of k

Asymptotic behavior in literature

References	(s,q)	Pointwise decay of metric
Christodoulou-Klainerman 93'	$(4,3)$	$r^{-\frac{3}{2}}$
Klainerman-Nicolò 03'	$(4,3)$	$r^{-\frac{3}{2}}$
Graf 20'	$(4,3)$	$r^{-\frac{3}{2}}$
Lindblad-Rodnianski 10'	$(3+\delta,\ 6)$	$r^{-1-\frac{\delta}{2}}$
Hintz-Vasy 17'	$(3+\delta,\ 26)$	$r^{-1-\frac{\delta}{2}}$
Ionescu-Pausader 19'	$(3-\delta,\ 200)$	$r^{-1+\frac{\delta}{2}}$
Lefloch-Ma 22'	$(2+\delta,\ 20)$	$r^{-\frac{1}{2}-\frac{\delta}{2}}$
Bieri 07'	$(2,2)$	$r^{-\frac{1}{2}}$
S. 23'	$(1+\delta,\ 2)$	$r^{-\frac{\delta}{2}}$

Main theorem

Theorem. [S. 23'] Let $s \in (1, 2]$ and let an initial data set (Σ_0, g, k) which is $(s, 2)$–asymptotically flat. Assume that all the quantities have size ε_0 in an initial layer region near Σ_0. Then, there exists a unique future development $(\mathcal{M}, \mathbf{g})$ satisfying the following properties:

- $(\mathcal{M}, \mathbf{g})$ can be foliated by a maximal-null foliation (C_u, Σ_t) whose outgoing leaves C_u are complete for all u;
- All connection coefficients Γ and curvature components R associated with the maximal-null foliation (C_u, Σ_t) have size ε_0.

Remark. In the particular case $s = 2$, we reobtain the result of Bieri

Minimal decay assumption $s > 1$

Asymptotic behaviors:

$$\mathbf{g} - \eta = o_{q+1}\left(r^{-\frac{s-1}{2}}\right), \quad \Gamma = o_q\left(r^{-\frac{s+1}{2}}\right), \quad R = o_{q-1}\left(r^{-\frac{s+3}{2}}\right)$$

Schematic null structure equations and Bianchi equations:

$$\nabla\!\!\!/_\mu \Gamma = R + \Gamma \cdot \Gamma, \qquad \nabla\!\!\!/_\mu R_{(1)} = \nabla\!\!\!/ R_{(2)} + \Gamma \cdot R$$

Nonlinear terms should have better order of decay than linear terms:

$$\frac{s+3}{2} < \frac{s+1}{2} + \frac{s+1}{2}, \qquad \frac{s+5}{2} < \frac{s+1}{2} + \frac{s+3}{2}$$

Both restrictions are equivalent to $s > 1$

Maximal foliation

$\mathcal{M}$ is foliated by maximal hypersurfaces Σ_t for $t \geqslant 0$ and

$$\mathbf{g} = -\phi^2 dt^2 + g_{ij} dx^i dx^j,$$

where ϕ is called the time lapse function

Second fundamental form:

$$k_{ij} := -\mathbf{g}(\mathbf{D}_{e_i} T, e_j),$$

where T denotes the unit future directed normal vectorfield on Σ_t

Electric-Magnetic decomposition:

$$E_{ij} := \mathbf{R}(T, e_i, T, e_j), \qquad H_{ij} := {}^*\mathbf{R}(T, e_i, T, e_j)$$

Maximal-null foliation

We fix a point $O \in \Sigma_0$. Emanating from O, the integral curve of T is denoted by $\varPhi$, called the symmetry axis

Given a radial foliation centered at O on Σ_0, we construct a forward outgoing null foliation $\{C_u\}_{u\in\mathbb{R}}$ emanating from $\varPhi \cup \Sigma_0$. The null cone emanating from O is denoted by C_0. We also denote

$$S(t,u) := C_u \cap \Sigma_t$$

Denote N the unit vectorfield tangent to Σ_t, oriented towards infinity and orthogonal to the leaves $S(t,u)$

Null frame: $e_4 = T + N$, $e_3 = T - N$ and e_1, e_2 tangent to $S(t,u)$

Principal quantities

Ricci coefficients:

$$\chi_{ab} := \mathbf{g}(\mathbf{D}_{e_a} e_4, e_b), \qquad \underline{\chi}_{ab} := \mathbf{g}(\mathbf{D}_{e_a} e_3, e_b), \qquad \xi_a := \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4} e_4, e_a),$$

$$\underline{\xi}_a := \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3} e_3, e_a), \qquad \omega := \frac{1}{4}\mathbf{g}(\mathbf{D}_{e_4} e_4, e_3), \qquad \underline{\omega} := \frac{1}{4}\mathbf{g}(\mathbf{D}_{e_3} e_3, e_4),$$

$$\eta_a := \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_3} e_4, e_a), \qquad \underline{\eta}_a := \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_4} e_3, e_a), \qquad \zeta_a := \frac{1}{2}\mathbf{g}(\mathbf{D}_{e_a} e_4, e_3)$$

Curvature components:

$$\alpha_{ab} := \mathbf{R}(e_a, e_4, e_b, e_4), \qquad \beta_a := \frac{1}{2}\mathbf{R}(e_a, e_4, e_3, e_4), \qquad \rho := \frac{1}{4}\mathbf{R}(e_3, e_4, e_3, e_4),$$

$$\underline{\alpha}_{ab} := \mathbf{R}(e_a, e_3, e_b, e_3), \qquad \underline{\beta}_a := \frac{1}{2}\mathbf{R}(e_a, e_3, e_3, e_4), \qquad \sigma := \frac{1}{4}{}^*\mathbf{R}(e_3, e_4, e_3, e_4)$$

Renormalized quantities

Expansion and shear:

$$\operatorname{tr}\chi := \mathbf{g}^{ab}\chi_{ab}, \qquad \widehat{\chi}_{ab} := \chi_{ab} - \frac{1}{2}\operatorname{tr}\chi\,\mathbf{g}_{ab},$$
$$\operatorname{tr}\underline{\chi} := \mathbf{g}^{ab}\underline{\chi}_{ab}, \qquad \underline{\widehat{\chi}}_{ab} := \underline{\chi}_{ab} - \frac{1}{2}\operatorname{tr}\underline{\chi}\,\mathbf{g}_{ab}$$

Renormalized curvature components:

$$\check{\rho} := \rho - \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}}, \qquad \check{\sigma} := \sigma - \frac{1}{2}\widehat{\chi}\wedge\underline{\widehat{\chi}}$$

Remark. The purpose of defining $\check{\rho}$ and $\check{\sigma}$ is to get rid of the most dangerous nonlinear terms of the Bianchi equations, i.e. $-\frac{1}{2}\widehat{\chi}\cdot\underline{\alpha}$ appearing in $\nabla\!\!\!/_3\rho$ and $-\frac{1}{2}\widehat{\chi}\wedge\underline{\alpha}$ appearing in $\nabla\!\!\!/_3\sigma$. This can be done according to $\nabla\!\!\!/_3\underline{\widehat{\chi}} = -\underline{\alpha} + \text{l.o.t.}$

Curvature estimates: r^p–weighted estimates

Area radius of $S(t,u)$ is defined by:

$$r := \sqrt{\frac{|S(t,u)|}{4\pi}}$$

Bianchi equations for (α,β)

$$\nabla\!\!\!/_3\alpha - \frac{1}{r}\alpha = -2 d\!\!\!/_2^*\beta + \mathbf{NL}, \tag{1}$$

$$\nabla\!\!\!/_4\beta + \frac{4}{r}\beta = d\!\!\!/_2\alpha + \mathbf{NL} \tag{2}$$

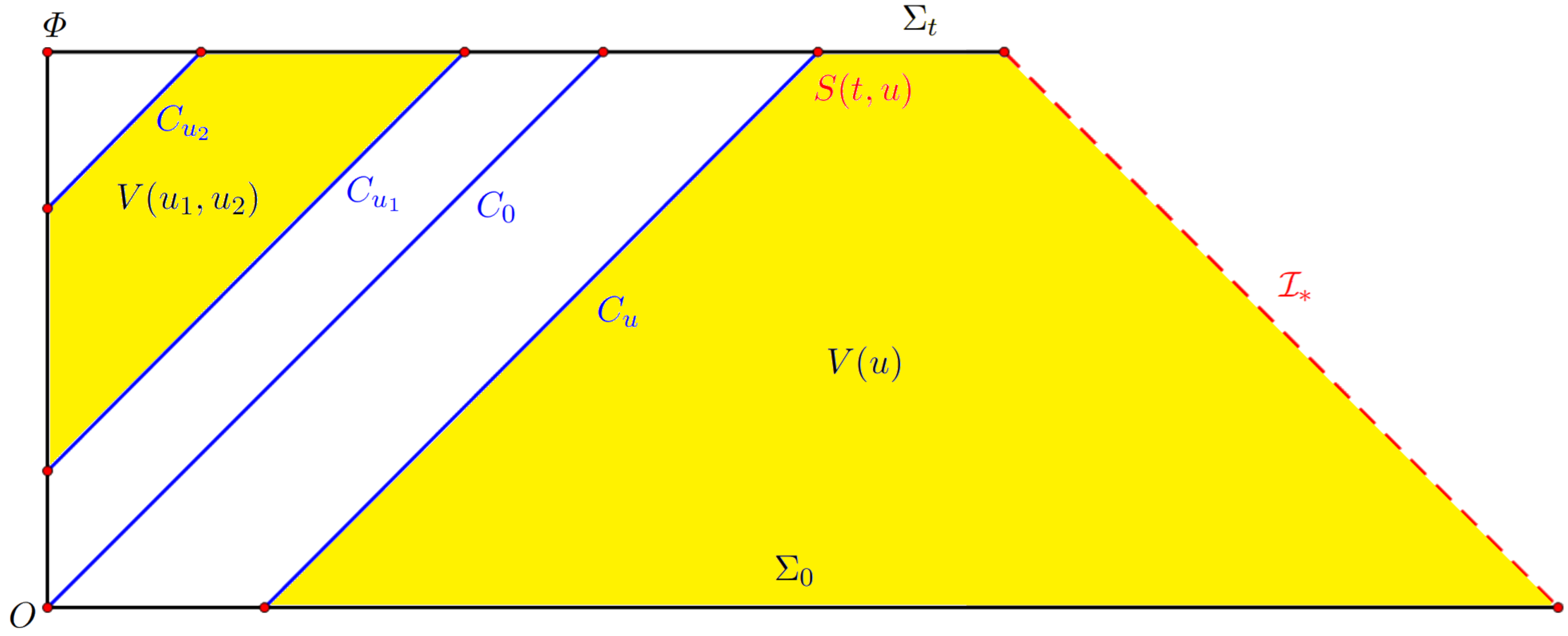
Summing $2r^p\alpha\times$(1) and $4r^p\beta\times$(2), we deduce

$$\mathbf{Div}(r^p|\alpha|^2 e_3) + 2\mathbf{Div}(r^p|\beta|^2 e_4) + pr^{p-1}|\alpha|^2 + 2(6-p)r^{p-1}|\beta|^2$$
$$= 4r^p \mathrm{d\!\!\!/iv}\,(\alpha\cdot\beta) + \mathbf{NL}$$

Curvature estimates in the exterior region

Taking $p = s$ and integrating it in $V = V(u)$, we obtain

$$\int_{C_u \cap V} r^s |\alpha|^2 + \int_{\Sigma_t \cap V} r^s (|\alpha|^2 + |\beta|^2) \lesssim \int_{\Sigma_0 \cap V} r^s (|\alpha|^2 + |\beta|^2) + \mathbf{NL}$$



Curvature estimates in the interior region

Taking $p \in (0, s]$ and integrating it in $V = V(u_1, u_2)$, we obtain

$$
\begin{aligned}
&\int_{C_{u_2} \cap V} r^p |\alpha|^2 + \int_{\Sigma_t \cap V} r^p (|\alpha|^2 + |\beta|^2) + \int_V r^{p-1} (|\alpha|^2 + |\beta|^2) \\
\lesssim &\int_{C_{u_1} \cap V} r^p |\alpha|^2 + \mathbf{NL}
\end{aligned}
$$

Remark. The integral along the symmetry axis Φ vanishes, i.e.

$$\int_{\Phi \cap V} r^p \left(|\alpha|^2 - |\beta|^2 \right) = 0$$

Remark. u–decay estimates in the interior region can be obtained by the standard mean value method of [Dafermos-Rodnianski 10']

Elliptic system for k

$3D$–elliptic equations:

$$\operatorname{tr} k = 0, \qquad \operatorname{div} k = 0, \qquad \operatorname{curl} k = H$$

Elliptic system for $i_Z k := Z \cdot k$:

$$\operatorname{div}(i_Z k) = \mathbf{NL}, \qquad \operatorname{curl}(i_Z k) = i_Z H + \mathbf{NL},$$

where $Z := rN$ denotes the position vectorfield

Commuting with $r^{\frac{s-2}{2}}$, we have:

$$\operatorname{div}(r^{\frac{s-2}{2}} i_Z k) = \frac{s-2}{2} r^{\frac{s-4}{2}} (i_Z k)_N + \mathbf{NL},$$

$$\operatorname{curl}(r^{\frac{s-2}{2}} i_Z k) = r^{\frac{s-2}{2}} i_Z H + \frac{s-2}{2} r^{\frac{s-4}{2}} \big(-(i_Z k)_2, (i_Z k)_1, 0 \big) + \mathbf{NL}$$

Elliptic estimates in $3D$–geometry

Proposition. Let ξ be a 1–form on Σ_t satisfying

$$\operatorname{div} \xi = D(\xi), \qquad \operatorname{curl} \xi = A(\xi)$$

Then, we have

$$\int_{\Sigma_t} |\nabla \xi|^2 = \int_{\Sigma_t} |D(\xi)|^2 + |A(\xi)|^2 - \operatorname{Ric}(\xi, \xi)$$

Hardy inequality. Let ξ be a 1–form on Σ_t, we have

$$\left(\frac{1}{4} - O(\varepsilon) \right) \int_{\Sigma_t} |\xi|^2 \leqslant \int_{\Sigma_t} |\nabla \xi|^2$$

Estimate for $i_Z k$

Taking $\xi = r^{\frac{s-2}{2}} i_Z k$, we obtain

$$\begin{aligned}\int_{\Sigma_t} |\nabla(r^{\frac{s-2}{2}} i_Z k)|^2 &\leqslant \int_{\Sigma_t} \frac{(s-2)^2}{4} r^{s-4} |(i_Z k)_N|^2 + C_{\delta_0} \int_{\Sigma_t} r^{s-2} |i_Z H|^2 \\ &+ \int_{\Sigma_t} \frac{(s-2)^2}{4} (1+\delta_0) r^{s-4} \left(|(i_Z k)_1|^2 + |(i_Z k)_2|^2 \right) + \mathbf{NL} \\ &\leqslant \frac{(s-2)^2(1+\delta_0)}{4} \int_{\Sigma_t} r^{s-4} |i_Z k|^2 + C_{\delta_0} \varepsilon_0^2\end{aligned}$$

Since $\frac{(s-2)^2}{4}(1+\delta_0) < \frac{1}{4} - O(\varepsilon)$, we have from Hardy inequality:

$$\int_{\Sigma_t} |\nabla(r^{\frac{s-2}{2}} i_Z k)|^2 + r^{s-4} |i_Z k|^2 \lesssim \varepsilon_0^2,$$

which implies the estimates for k_{NN} and k_{aN}

Estimate for k_{ab}

Recall that

$$\begin{aligned}
\not{\mathrm{d}}\mathrm{iv}\,\widehat{\kappa} &= \frac{1}{2}(-\underline{\beta} + \beta) - \frac{1}{2}\not\nabla\delta - \frac{1}{r}\epsilon + \mathbf{NL},\\
\not\nabla_N\widehat{\kappa} + \frac{1}{r}\widehat{\kappa} &= \frac{1}{4}(-\underline{\alpha} + \alpha) + \frac{1}{2}\not\nabla\widehat{\otimes}\epsilon + \mathbf{NL},\\
\mathrm{tr}\,\kappa &= -\delta,
\end{aligned}$$

where we denoted

$$\kappa_{ab} := k_{ab}, \qquad \epsilon_a := k_{aN}, \qquad \delta := k_{NN}$$

The estimate for κ follows directly from that of ϵ and δ

Time lapse function estimate

Denoting $\varphi := \log \phi$, we have from $\Delta\phi = |k|^2\phi$ that

$$\Delta\varphi = |\nabla\varphi|^2 - |k|^2 = \mathbf{NL}$$

We compute

$$\begin{aligned} r^{s-2}|\nabla\varphi|^2 &= \nabla^i(r^{s-2}\varphi\nabla_i\varphi) - \nabla^i(r^{s-2})\varphi\nabla_i\varphi - r^{s-2}\varphi\Delta\varphi \\ &= \mathrm{div}(r^{s-2}\varphi\nabla\varphi) - (s-2)r^{s-3}\varphi N(\varphi) + \mathbf{NL} \end{aligned}$$

We also have

$$\int_{\Sigma_t} 2r^{s-3}\varphi N(\varphi) = -(s-1)\int_{\Sigma_t} r^{s-4}|\varphi|^2 + \mathbf{NL}$$

Thus, we obtain for $1 < s \leqslant 2$

$$\int_{\Sigma_t} r^{s-2}|\nabla\varphi|^2 = \frac{(s-1)(s-2)}{2}\int_{\Sigma_t} r^{s-4}|\varphi|^2 + \mathbf{NL} \leqslant \mathbf{NL} \lesssim \varepsilon_0^2$$

Identities for connection coefficients

The following identities holds for a maximal-null foliation:

$$\chi_{ab} = \theta_{ab} - \kappa_{ab}, \qquad \underline{\chi}_{ab} = -\theta_{ab} - \kappa_{ab},$$

$$\xi_a = 0, \qquad \underline{\xi}_a = \nabla\!\!\!/_a \log\phi - \nabla\!\!\!/_a \log a,$$

$$\eta_a = \nabla\!\!\!/_a \log a + \epsilon_a, \qquad \underline{\eta}_a = \nabla\!\!\!/_a \log\phi - \epsilon_a,$$

$$2\omega = -\nabla_N \log\phi + \delta, \qquad 2\underline{\omega} = \nabla_N \log\phi + \delta,$$

$$\zeta_a = \epsilon_a,$$

where

$$a := |\nabla u|^{-1}, \qquad \theta_{ab} := \mathbf{g}(\mathbf{D}_{e_a} N, e_b)$$

All the Ricci coefficients are linear combinations of k, $\nabla \log\phi$, χ_{ab} and η_a. It thus remains to control χ_{ab} and η_a

Null connection estimates

Recall that

$$\nabla\!\!\!/_4 \mathrm{tr}\,\chi + \frac{1}{2}(\mathrm{tr}\,\chi)^2 = -2\omega\,\mathrm{tr}\,\chi + \mathbf{NL},$$

$$\mathrm{d}\!\!/\mathrm{iv}\,\widehat{\chi} = \frac{1}{2}\nabla\!\!\!/\mathrm{tr}\,\chi + \frac{1}{2}\mathrm{tr}\,\chi\,\zeta - \beta + \mathbf{NL},$$

$$\nabla\!\!\!/_4 \eta + \frac{1}{2}\mathrm{tr}\,\chi\,\eta = \frac{1}{2}\mathrm{tr}\,\chi\,\underline{\eta} - \beta + \mathbf{NL}$$

Thus, $\mathrm{tr}\,\chi - \frac{2}{r}$, $\nabla\!\!\!/\mathrm{tr}\,\chi$, $\widehat{\chi}$ and η can be estimated one by one by their values on Σ_0 and $\varPhi$. Finally, we deduce

$$\|R\| \lesssim \varepsilon_0, \qquad \|\Gamma\| \lesssim \varepsilon_0$$

Conclusion: global stability of Minkowski holds true for $(s,2)$–asymptotically flat initial data where $s \in (1,2]$

Thanks for your attention!