Counterexamples to Unique Continuation for Critically Singular Waves

Applications to Anti-de Sitter Spacetimes

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Introduction

Big picture I

Problem

Study non-uniqueness of solutions to:

$$\mathcal{P}u := \left[\Box_g + \frac{\xi(\sigma, y)}{\sigma^2}\right] u = 0, \tag{1}$$

on $\Omega := (0, \sigma_0) \times \mathcal{I}$, $\mathcal{I} \subset \mathbb{R}^d$ *open, with* g, ξ *smooth.*

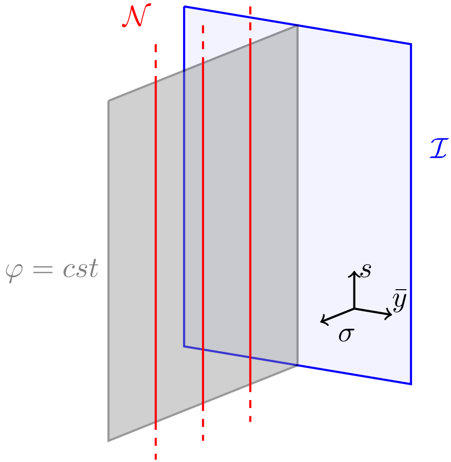
Remark

$\Box_g$ *and* $\xi \cdot \sigma^{-2}$ *have the same scaling*
$\Rightarrow$ *well-posedness, decay, ...*

- When does Unique Continuation (**UC**) fail for $\mathcal{P}$ from $\{0\} \times \mathcal{I}$?

Big picture II

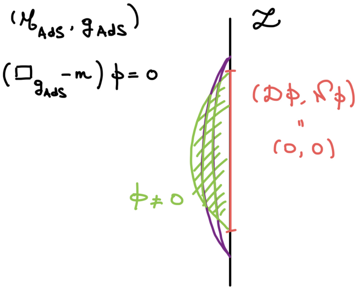
- Existence of trapped null geodesics $\mathcal{N} \Rightarrow$ Failure of **UC**.



Big picture III

Motivation (AdS/CFT correspondence)

Potential mechanism for counterexamples to the AdS/CFT correspondence.



- Show failure of UC for Klein-Gordon equations from conformal boundary.

Background: unique continuation problems

- $\mathcal{L}$ linear differential operator, $\mathcal{U} \subset \mathbb{R}^{d+1}$ and $\Sigma \subset \mathcal{U}$ a hyperplane.

Problem (Unique Continuation)

*If $\mathcal{L}u = 0$ in $\mathcal{U}$, does Cauchy data on Σ **uniquely** determine u on one side of Σ?*

$$\Updownarrow$$

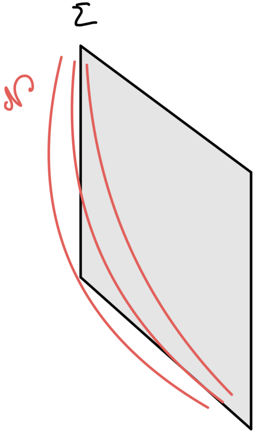
$\mathcal{L}u = 0$ in $\mathcal{U}$ and $(\mathcal{D}u, \mathcal{N}u)|_\Sigma = 0$, then $u \equiv 0$ on one side of Σ?

- **Holmgren:** $\mathcal{L}$ real-analytic coefficients, Σ analytic non-characteristic $\Rightarrow$ UC holds.
- **Calderòn, Carleman, Hörmander:** extends to $\mathcal{L}$ smooth $\Rightarrow$ requires (strong) pseudoconvexity.

Background: failure of pseudoconvexity

Remark

Pseudoconvexity forbids the existence of trapped null geodesics near Σ.



Alinhac-Baouendi's counterexamples

Theorem (Alinhac-Baouendi (informal))

- *Let $x_0 \in \Sigma$ and let $\mathcal{L}$ be a differential operator of order m with* ***smooth and bounded*** *coefficients near x_0.*
- *Assume there exists a family $\mathcal{N}$ of trapped null bicharacteristics near x_0*

Then,

- *There exist functions u, V complex-valued, smooth and bounded on Ω_0, a* ***neighbourhood*** *of x_0, satisfying:*

$$(\mathcal{L} - V)u = 0,$$

- *u, V vanish at infinite order on $\Omega_0 \cap \Sigma$.*
- *supp $u = \overline{\Omega}_0 \cap \Sigma$.*

Alinhac-Baouendi – Limitations

Problem

- *Potential V generally necessary*

$$\Updownarrow$$

0th-order perturbation to Holmgren's theorem.

- *Linearity crucial (Métivier's counterexamples)*
- *Functions u, V **complex-valued**.*
- ***Local** counterexamples $\Rightarrow$ do they persist along $\mathcal{N}$?*
- *Operators with **unbounded** coefficients?*

Singular wave operator

$$\Omega := (0, \sigma_0)_\sigma \times \mathcal{I}_y \subset \mathbb{R}^{d+1},$$

with $\mathcal{I} = (s_-, s_+)_s \times \mathcal{I}'_{\bar{y}} \subset \mathbb{R}^d$ open.

Consider:

$$\mathcal{P} := \Box_g + \xi(\sigma, y) \cdot \sigma^{-2},$$

with $g \in \mathcal{B}^\infty(\Omega; \mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ Lorentzian and $\xi \in \mathcal{B}^\infty(\Omega; \mathbb{C})$.

Remark

$\Box_g$ *and* $\xi \cdot \sigma^{-2}$ *have the same scaling* $\Rightarrow$ ***cannot treat perturbatively****!*

Asymptotics of solutions

Presence of singular term **radically** alters the asymptotics of solutions!

Example (3+1 Einstein cylinder)

$$\left[\Box_g - \frac{\lambda}{\rho^2}\right]\psi = 0, \qquad g = -dt^2 + d\rho^2 + \cos^2\rho \cdot \not{g}_{\mathbb{S}^2}(\theta,\phi),$$

implies the expansion, near $\{\rho = 0\}$:

$$\begin{aligned}\psi(t,\rho,\theta,\phi) =& \rho^{\lambda_+}\left(\psi_+(t,\theta,\phi) + O(\rho^2)\right)\\ &+ \rho^{\lambda_-}\left(\psi_-(t,\theta,\phi) + O(\rho^2)\right),\end{aligned}$$

with $\lambda_\pm = 1/2 \pm \sqrt{1/4+\lambda}$, and (ψ_-,ψ_+) Dirichlet/Neumann branches.

Asymptotically Anti-de Sitter spacetimes

Definition (Asymptotically Anti-de Sitter)

Manifold $(\mathcal{M}, g)$ of the form:

$$\mathcal{M} := (0, \rho_0)_\rho \times \mathcal{I}_x, \qquad g(\rho, x) := \rho^{-2} \left(d\rho^2 + \mathsf{g}(\rho, x) \right),$$

solution to $\mathrm{Ric}(g) = -d \cdot g$.

Remark (Singular operators in AdS)

$\mathcal{P} := \Box_g - m$ ***conformally equivalent*** *to:*

$$\bar{\mathcal{P}} := \Box_h - \rho^{-2} \left(m + \frac{d^2 - 1}{4} \right) + V,$$

with $h := \rho^2 g$, V *smooth.*

Klein-Gordon in AdS

Remark

Linearised wave equation for Weyl curvature W on $(\mathcal{M}, g)$ decomposes into:

$$(\Box_g - m_\Psi)\Psi = \text{l.o.t.},$$

with Ψ non-trivial components of W.

Example (3+1 gravity)

W decomposes into (Ψ_1, Ψ_2) with masses:

$$(m_1, m_2) = (-2, -2).$$

Remark

Klein-Gordon ***conformally invariant*** *if*

$$m = m_c := -\frac{d^2 - 1}{4}.$$

AdS/CFT correspondence

Definition (Conformal boundary)

$(\mathcal{I}, \mathfrak{g})$ **conformal boundary**, with $\mathfrak{g} := \mathrm{g}(0^+, x)$.

Remark

$\mathcal{I}$ of timelike nature.

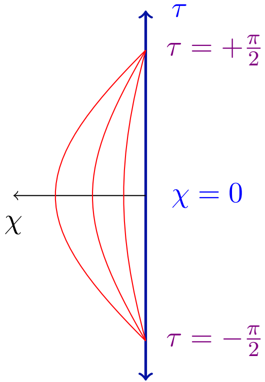
AdS/CFT correspondence

Dynamical one-to-one correspondence between interior of AdS and conformal boundary

$\Downarrow$ **(linearised)**

UC problem for Klein-Gordon from conformal boundary $(\mathcal{I}, \mathfrak{g})$.

Trapped Null Geodesics and AdS



Result

Statement

Let $\Omega := (0,\sigma_0)_\sigma \times \mathcal{I} \subset \mathbb{R}^{d+1}$ with $\mathcal{I} := (s_-, s_+)_s \times \mathcal{I}'$ open let $\mathcal{P}$ be as in (1).

Theorem (G., Shao '23)

Assume there exist:

1. $C > 0$, $\gamma \geq 0$ *such that* $g^{-1}(d\sigma, d\sigma) \geq C\sigma^\gamma$
2. *a smooth and bounded function* φ *satisfying:*

$$g^{-1}(d\varphi, d\varphi) = 0, \qquad 2\mathrm{grad}_g\, \varphi = \partial_s$$

Then, there exist $u, a \in \mathcal{B}^\infty(\Omega; \mathbb{C})$ *such that:*

3. *All derivatives of* u, a *vanish faster than* σ^N *as* $\sigma \searrow 0$, *for all* $N \geq 0$.
3. *u is supported on* Ω
3. u, a *satisfy on* Ω*:*

$$(\mathcal{P} - a)u = 0.$$

Discussion

1. Solution u **counterexample to UC** for $\mathcal{P}-a$ from $\{0\} \times \mathcal{I}$, where a seen as a perturbation:

$$a = O(\sigma^N), \qquad \text{for all } N \geq 0, \text{ as } \sigma \searrow 0.$$

2. Functions u, a **inherently** complex-valued.
3. Condition 1 allows $\mathcal{I}$ either asymptotically null ($\gamma > 0$) or timelike ($\gamma = 0$).
4. First condition in 2 $\Rightarrow$ integral curves of $\mathrm{grad}_g \varphi$ generate family of null geodesics $\mathcal{N}$.
5. Second condition: choice of gauge such that geodesics in $\mathcal{N}$

$$s \mapsto \gamma(s) := (\sigma_\star, \bar{y}_\star, s), \qquad (\sigma_\star, \bar{y}_\star) \in (0, \sigma_0) \times \mathcal{I}'. \qquad (2)$$

Sketch of construction

Linear Geometric Optics – Basics

Idea (Alinhac-Baouendi)

Propagate along $\mathcal{N}$ high-frequency approximate solutions and sum them appropriately.

Remark (Basic Geometric Optics Approximation)

Define $\psi := e^{i\lambda\varphi} b$, $b = \sum_{k=0}^{N} b_k \lambda^{-k}$, *with* φ *satisfying assumptions* 1, 2 *and* (b_k) *solving:*

$$\begin{cases} (\text{grad}_g \varphi + \Box_g \varphi) b_0 = 0 \\ (\text{grad}_g \varphi + \Box_g \varphi) b_k + \mathcal{P} b_{k-1} = 0, \qquad 1 \leq k \leq N. \end{cases}$$

Then, ψ *is an approximate solution in the following sense:*

$$\mathcal{P}\psi = e^{i\lambda\varphi} \lambda^{-N} \mathcal{P} b_N. \tag{3}$$

Construction – Generalities

Remark (Main idea, Alinhac-Baouendi)

Construct (c_j) geometric optics coefficients such that:

$$u := e^{i\lambda\varphi}\sum_{j=0}^{N} c_j\lambda^{-j}, \qquad a := \frac{\mathcal{P}u}{u},$$

smooth and bounded and vanishing at infinite order as $\sigma \searrow 0$.

To control amplitudes of (u, a):

- Localise around bands $\Omega_n \sim n^{-1}$ of width $\sim n^{-2}$ such that $\Omega_n \cap \Omega_m = \emptyset$ whenever $|m - n| > 1$.
- Complex φ such that $\operatorname{Im}\varphi \geq 0$.
- Sum each geometric optics band $u = \sum_{n\geq n_0} v_n$ with v_n localised around Ω_n.
- Study where $|v_n| = |v_{n\pm1}| \Rightarrow$ potential blow-up of a!

First solutions

- Let $(c_{j,n})_{j\geq 0}$ solving:

$$\begin{cases} T_1 c_{0,n} = 0, & \underbrace{T_1 c_{j,n}}_{\text{Regular transport}} + n^{-\alpha} \underbrace{T_{2,n} c_{j-1,n}}_{\text{Singular}} = 0, \quad j \geq 1 \\ (c_{0,n}, c_{j,n})|_{s=0} = (\underline{\chi}_n, 0), & \end{cases}$$

Definition

For any $n \geq n_0$, we define the following functions on Ω:

$$c_{n,\star} := \sum_{j=1}^{\lfloor I_n/3 \rfloor} n^{-j\alpha} c_{n,j},$$

$$v_n := e^{in^{2\alpha}\varphi} e^{n^2 f_n} \left(c_{n,0} + c_{n,\star} \right).$$

$(I_n)_n \subset \mathbb{N}$ strictly increasing sequence necessary $\Rightarrow$ uniform estimates in j, n.

Destructive interferences

Problem

Since supp $v_n \cap$ *supp* $v_{n\pm1} \neq \emptyset$*, the function:*

$$\underline{a} := \frac{\mathcal{P}\left(\sum_n v_n\right)}{\sum_n v_n}$$

may blow-up on $S_{n,\pm} := \{|v_n| = |v_{n\pm1}|\}$*.*

Proposition

If $n \geq n_0$*, then* $S_{n,\pm}$ *is a smooth graph in* $\Omega_n \cap \Omega_{n+1}$ *of the form*

$$S_{n,\pm} = \{(\sigma, y) \in \Omega_n \cap \Omega_{n\pm1} \mid y \in \mathcal{I},\, \sigma = \mathfrak{s}_n^{\pm}(y)\},$$

where $\mathfrak{s}_n^{\pm} \sim n^{-2\alpha}$ *some appropriate smooth function.*

$\Rightarrow$ **Does not make use of the implicit function theorem!**

Modification of the bands

Idea (Alinhac-Baouendi, modified)

Modify v_n by a function ω_n satisfying $\omega_n|_{S_{n,\pm}} = \partial_\sigma \omega_n|_{S_{n,\pm}} = 0$ to make $\underline{a}_n$ and its derivatives vanish (here, a finite number of times) on $S_{n,\pm}$.

- Interpolating ω_n between the two hyperplanes $S_{n,\pm}$. **Could not be done using Whitney extension theorem**!

Definition

For any $n \geq n_0$, define:

$$\tilde{v}_n := e^{in^{2\alpha}\varphi} e^{n^2 f_n} \left(c_{n,0} + c_{n,\star} + \omega_n \right),$$

$$u := \sum_{n \geq n_0} \tilde{v}_n, \qquad a := \frac{\mathcal{P}u}{u},$$

Gluing the pieces together

Proposition

$u \in C^\infty(\Omega)$, and the following holds for any $N, \mu > 0$:

$$\lim_{\sigma_0 \searrow 0} \sup_{\{\sigma=\sigma_0\}} \left|\sigma^{-\mu} D^N u\right| = 0.$$

Proposition

a is a smooth function in a neighbourhood of $\{\sigma = 0\}$ in Ω. Furthermore,

$$\lim_{\sigma_0 \searrow 0} \sup_{\{\sigma=\sigma_0\}} \left|\sigma^{-\mu} D^N a\right| = 0, \qquad N, \mu > 0.$$

Applications to Anti-de Sitter

Anti-de Sitter spacetimes

Planar Anti-de Sitter

$$\mathcal{M}_{plan} := (0,\infty)_r \times \mathbb{R}^d_x, \qquad g_{plan} := r^{-2}dr^2 + r^2\eta,$$

with η Minkowski metric on $\mathbb{R}^d$.

Pure Anti-de Sitter

$$\mathcal{M}_{AdS} := \mathbb{R}_\tau \times (0,\infty)_r \times \mathbb{S}^{d-1}_\omega,$$
$$g_{AdS} := -(1+r^2)d\tau^2 + (1+r^2)^{-1}dr^2 + r^2\not{g}(\omega),$$

with $\not{g}$ metric on $\mathbb{S}^{d-1}$.

Conformal compactification

Conformal compactification of Planar AdS

Defining $\rho := r^{-1}$, conformal isometry:

$$\bar{\mathcal{M}}_{plan} := (0, +\infty)_\rho \times \mathbb{R}^d, \qquad \bar{g}_{plan} := d\rho^2 + \eta,$$

with conformal factor $\Omega^2 = \rho^2$.

Conformal compactification of pure AdS

Defining $\chi := \pi/2 - \arctan r$, conformal isometry:

$$\bar{\mathcal{M}}_{AdS} := \mathbb{R}_\tau \times (0, \pi/2)_\chi \times \mathbb{S}^{d-1}_\omega,$$
$$\bar{g}_{AdS} := -d\tau^2 + d\chi^2 + \cos^2\chi \cdot \not{g}(\omega),$$

with conformal factor $\Omega^2 = \sin^2\chi$.

Planar Anti-de Sitter

Consider $(\mathcal{M}_{plan}, g_{plan})$ and fix $m \in \mathbb{R}$.

Corollary (Counterexamples for Planar AdS)

For any $t_- < t < t_+$, there exists V smooth and bounded and a smooth u counterexample to UC for:

$$(\Box_{g_{plan}} - m)u = Vu,$$

defined on the timespan $\Omega := \{t_- < t < t_+\}$. Furthermore, $u \in \mathcal{B}^\infty(\Omega; \mathbb{C})$.

Problem

The theorem provides counterexamples supported on the whole $\{t_- < t < t_+\} \cap \{r_0 < r\} \Rightarrow u$ may fail to be in $L^2(\Omega)$!

Deformation of $\{\sigma = 0\}$

Remark (Potential solutions)

- *Construct counterexamples on $\tilde{\Omega} \subset \Omega$ bounded and smoothly zero-extend on Ω*

$\Rightarrow$ *not necessarily a counterexample on $\Omega \setminus \tilde{\Omega}$.*

- *Apply an appropriate cut-off χ such that $\chi \equiv 1$ on $\tilde{\Omega}$.*

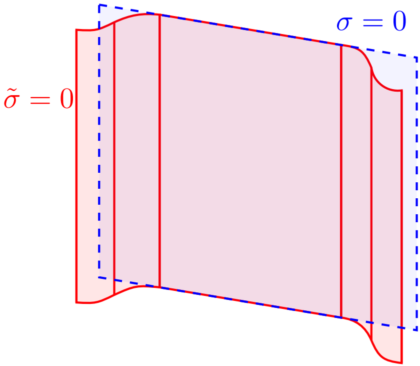
$\Rightarrow$ *have to deal with $\chi^{-1} u^{-1} [\mathcal{P}, \chi] u$.*

Idea:

Deform level sets of σ by $\tilde{\sigma}$ such that level sets of $\{\sigma = 0\}$ and $\{\tilde{\sigma} = 0\}$ agree on a **bounded** set.

$\Rightarrow$ Construct counterexamples using $(\tilde{\sigma}, s, \bar{y})$.

Deformation of $\{\sigma = 0\}$



Remark

Level sets of σ and $\tilde{\sigma}$ still generated by the same family of null geodesics!

Pure Anti-de Sitter

Consider $(\mathcal{M}_{AdS}, g_{AdS})$ and fix $m \in \mathbb{R}$.

Corollary

For any $\tau_- < \tau_+$ satisfying $\tau_+ - \tau_- < \pi$, there exist V smooth and bounded, and a smooth counterexample u for:

$$(\Box_{g_{AdS}} - m)u = Vu,$$

defined on the timespan $\Omega := \{\tau_- < \tau < \tau_+\}$. Furthermore, $u \in \mathcal{B}^\infty(\Omega; \mathbb{C})$.

Remark

*$\mathcal{I}$ has compact cross-sections of positive curvature $\Rightarrow$ **hairy ball theorem** (no global direction of propagation for d even)*

Holzegel-Shao ('16), McGill-Shao ('20): **uniqueness** for data prescribed on $\{\tau_- < \tau < \tau_+\}$ as long as $\tau_+ - \tau_- > \pi$.

Construction on pure AdS

1. Fix $\varepsilon > 0$ small, define the following region:

$$\mathcal{M}_{P,\varepsilon} := \left\{|\tau| \leq \pi/2 - \varepsilon,\, \omega^d < 0\right\} \subseteq \mathcal{M}_{AdS},$$

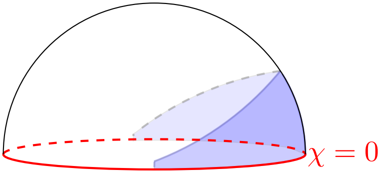
2. $(\mathcal{M}_{P,\varepsilon}, g_{AdS})$ isometrically embeds into a portion of $(\mathcal{M}_{plan}, g_{plan})$ (Poincaré patch)
3. Obtain a counterexample

$u \in C^\infty\left(\mathcal{M}_{plan} \cap \{\rho < \rho_0\} \cap \{|t| \leq c\varepsilon^{-1}\}\right)$ with:

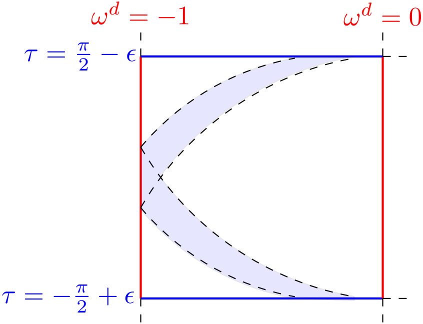
$$B_{\delta/2} \subset \operatorname{supp} u \subset B_\delta, \qquad \delta \ll 1.$$

$$B_\delta := (\mathcal{M}_{plan} \cap \{\rho < \rho_0\} \cap \{|t| \leq c\varepsilon^{-1}\}) \cap \{|\bar{x} - t\bar{k}| \leq \delta\varepsilon\},\ \left|\bar{k}\right| = 1.$$

Domain & support of u



(a) Domain of u in $\mathcal{M}_{P,\epsilon}$ at a fixed time τ.



(b) Support of u projected on the $\tau - \omega^d$ plane.

Alinhac-Baouendi's contruction

- Consider instead the complex geometric approximation:

$$e^{-i(\delta^{-2\alpha}\varphi+i\delta^{-2}\sigma)}\tilde{\mathcal{P}}e^{i(\delta^{-2\alpha}\varphi+i\delta^{-2}\sigma)}, \quad 0<\delta\ll 1, \quad \sigma = O_{\delta\searrow 0}(1)$$

- Let $(c_k)_{k\geq 0}$ be a sequence of functions on $\Omega\times(0,\delta_0)$ solving the system:

$$\begin{cases} T_1 c_0 = 0, \qquad \underbrace{T_1 c_i}_{\textcolor{red}{Transport}} + \delta^\alpha \underbrace{T_2 c_{i-1}}_{\textcolor{red}{Inhomogeneous}} = 0, \\ (c_0, c_i)|_\Sigma = (1,0), \end{cases}$$

in some neighbourhood $\Omega_0\times(0,\delta_0)$ such that:

$$\inf_{\Omega_0}|c_0|\geq 1/2, \qquad c_i = d_i|_{z=\delta^{-2}(\sigma-\delta)}, \tag{4}$$

with $d_i\in\mathcal{B}^\infty(\Omega_0\times(0,\delta_0)\times\mathbb{R}_z)$.

Alinhac-Baouendi's contruction (First solution)

Remark

Possible only if α big enough and the coefficients of $\tilde{\mathcal{P}}$ are smooth and bounded.

- Define:

$$v(x,\delta) = e^{i\tau^{2\alpha}\varphi} e^{\delta^{-2}\sigma} c, \quad c = c_0 + \sum_{i\geq 1} c_i \delta^{i\alpha} \chi\left(\delta^{\alpha}\epsilon_j^{-1}\right), \quad (5)$$

  with χ an appropriate cut-off and $(\epsilon_j)_j$ a sequence of positive numbers.

- Show that there exists a smooth function r satisfying:

$$\tilde{\mathcal{P}} v = r v, \quad (6)$$

  in $\Omega \times (0,\delta_0)$ with $\partial_x^M \partial_\delta^m r = O_{\delta\searrow 0}(\delta^N)$, for all $M, m, N \geq 0$.

Alinhac-Baouendi's contruction (discretisation)

- Define the functions v_k in Ω_0 to be given by:

$$v_k(x) = v(x, k^{-1}), \qquad k \geq k_0,$$

for some k_0 in $\mathbb{N}$.

Idea

Construct $u = \sum_{k \geq k_0} v_k$.

Problem: The function

$$a := \frac{\tilde{\mathcal{P}} \sum_k v_k}{\sum_k v_k} \tag{7}$$

is not well-defined where $\{|v_k| = |v_{k\pm1}|\}$.

Modification of the solutions

- Characterise the set S where $\{|v|(x,\delta) = |v|(x, \frac{\delta}{1-\delta})\}$, smooth hypersurface:

$$S = \{\sigma = \xi(y,\delta) := \delta - \frac{2}{3}\delta^3 + O_y(\delta^3)\} \cap \Omega_0,$$

where Ω_0 is a neighbourhood of x_0.

- Construct ω a function satisfying $\omega|_S = \partial_\sigma \omega|_S = 0$ and such that:

$$a = \frac{\tilde{\mathcal{P}}(v + e^{i\Psi} c_0 \omega)}{v + e^{i\Psi} c_0 \omega} \tag{8}$$

vanishes at infinite order on $\{\delta = 0\}$ and S.

- Construction relies on Whitney's extension theorem.
- Requires one to uniformly control an arbitrary number of derivatives in order to obtain ω smooth.

Gluing the pieces together

- Define $\tilde{v}_k := (v + e^{i\Psi} c_0 \omega)|_{\delta = k^{-1}} \cdot \tilde{\chi}_k$, with some appropriate cutoff $\tilde{\chi}_k$ such that:

$$\text{supp } \tilde{v}_k = \{\sigma \sim k^{-1} + O(k^{-2})\}$$

- Define $u := \sum_{k \geq k_0} \tilde{v}_k$ and show that both u and:

$$a := \frac{\tilde{\mathcal{P}} u}{u}, \tag{9}$$

have smooth and bounded derivative and vanish at infinite order on $\{\sigma = 0\} \cap \tilde{\Omega}_0$, where $\tilde{\Omega}_0$ is some neighbourhood of x_0.