Stability of Gravitational Collapse

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Newtonian stars

Classical model of a star: sphere of gas under Newtonian gravity.

- Balance between pressure and gravity in a static star;
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- Balance between pressure and gravity in a static star;
- As gas burns, balance shifts;
- Possible collapse? Supernova?

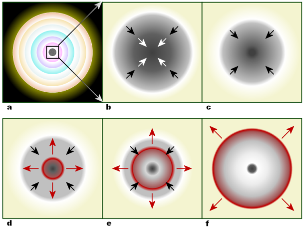

Figure: Image credit: R.J. Hall

Euler-Poisson equations

Euler-Poisson equations (gas dynamics with Newtonian gravity):

$$
\begin{cases}
\partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \\
\rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\big) + \nabla_{\mathbf{x}} p(\rho) = -\rho \nabla \Phi, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \\
\Delta \Phi = 4\pi\rho, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3.
\end{cases}
$$

ρ is density, $\mathbf{u}$ is velocity, p is pressure, Φ is gravitational potential.
We assume the equation of state

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p = p(\rho) = \rho^\gamma, \quad \gamma \in \Big(1, \frac{4}{3}\Big).
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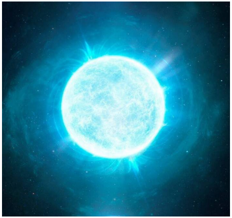

Example adiabatic exponents
$\gamma = \frac{5}{3}$ – monatomic gas, used for fully convective star cores (e.g. red giants);
$\gamma = \frac{4}{3}$ – high mass white dwarf stars, main-sequence stars (e.g. the Sun).

Collapse

Collapse is the formation of a *singularity* at the origin, i.e.

$$\rho(t,0) \to \infty \quad \text{as} \quad t \to 0- \, .$$

- For $\gamma > \frac{4}{3}$, no finite mass and energy collapse possible.
- For $\gamma = \frac{4}{3}$, Goldreich–Weber collapse - unsuitable model for outer core.

Similarity Hypothesis

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Hypothesis

On approach to singularity, isolated stars generically adopt an (approximately) self-similar form.

- Astrophysical literature: Harada, Maeda, Ori, Piran, Gundlach,...
- Requires full nonlinear stability
- Newtonian/relativistic gravity

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Key Features:

- Non-linearity;
- Intertwining of spatial and time scales;
- Good initial data leads to badly behaved solutions!

Scaling and Criticality

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Scaling

Let $\rho = \rho(t,r)$, $\mathbf{u} = u(t,r)\frac{\mathbf{x}}{|\mathbf{x}|}$, $r = |\mathbf{x}|$, solve Euler-Poisson, $\lambda > 0$. Then

$$\rho_\lambda(t,r) = \lambda^{-\frac{2}{2-\gamma}}\rho\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big), \quad u_\lambda(t,r) = \lambda^{-\frac{\gamma-1}{2-\gamma}}u\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big)$$

is also a solution. (NB: This is a *unique* scaling!)

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Natural notions of mass and energy for Euler-Poisson:

$$M[\rho] = \int_0^\infty \rho\, r^2 \mathrm{d}r, \quad E[\rho,u] = \int_0^\infty \left(\frac{1}{2}\rho u^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}\rho\Phi\right) r^2 \mathrm{d}r.$$

Under scaling,

$$M[\rho_\lambda] = \lambda^{\frac{4-3\gamma}{2-\gamma}} M[\rho], \quad E[\rho_\lambda, u_\lambda] = \lambda^{\frac{6-5\gamma}{2-\gamma}} E[\rho,u].$$

Thus $\gamma = \frac{4}{3}$ is *mass-critical*, $\gamma = \frac{6}{5}$ is *energy-critical*.

Existing results I

Classical and numerical work

- Taylor, Von Neumann, Sedov, Güderley '40s: study implosion and explosion for Euler equations;
- Larson–Penston '69: numerical solution for $\gamma = 1$;
- Hunter '77: family of numerical solutions for $\gamma = 1$;
- Yahil '83: numerical solutions for $\gamma \in [\frac{6}{5}, \frac{4}{3})$;
- Maeda–Harada '01: numerical evidence towards mode stability of Larson–Penston;
- Luo–Shi '14: numerical solutions for non-isentropic dynamics with $\gamma > \frac{4}{3}$.

Existing results II

Rigorous works

- Merle–Raphaël–Rodnianski–Szeftel ’22: existence of a imploding self-similar solutions for Euler;
- Guo–Hadžić–Jang ’21: construction of LP solution;
- Guo–Hadžić–Jang ’23: construction of relativistic analogue of LP;
- Alexander–Hadžić–S. ’23: existence of non-isentropic collapse solutions with $\gamma > \frac{4}{3}$;
- Sandine ’23: existence of a sub-family of highly oscillatory Hunter solutions;
- Jang–Liu–S. ’23, ’24: existence of Guderley solution

Main Results

Self-similar ansatz: let $y = r(-t)^{-(2-\gamma)}$,

$$\rho(t,r) = (-t)^{-2}\tilde{\rho}(y), \quad u(t,r) = (-t)^{1-\gamma}\tilde{u}(y).$$

Theorem (Guo–Hadžić–Jang–S. '22)

For all $\gamma \in [1, \frac{4}{3})$, there exists a smooth, self-similar collapse solution to the Euler-Poisson system satisfying the sign and monotonicity properties for $y > 0$,

$$\tilde{u}(y) < 0 < \tilde{\rho}(y), \qquad \tilde{\rho}'(y) < 0 < \left(\frac{\tilde{u}(y)}{y}\right)'.$$

NB: Existence for $\gamma = 1$ due to Guo–Hadžić–Jang '21

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Theorem (Guo–Hadžić–Jang–S. '24)

The Larson-Penston solution ($\gamma = 1$) is nonlinearly stable in the class of radially symmetric solutions.

Existence: ODE system

Defining a convenient variable $\omega(y) = \tilde{u}(y)/y + 2 - \gamma$, self-similar Euler-Poisson becomes

$$\tilde{\rho}' = \frac{y\tilde{\rho}h(\tilde{\rho},\omega)}{\gamma\tilde{\rho}^{\gamma-1} - y^2\omega^2},$$
$$\omega' = \frac{4 - 3\gamma - 3\omega}{y} - \frac{y\omega h(\tilde{\rho},\omega)}{\gamma\tilde{\rho}^{\gamma-1} - y^2\omega^2},$$

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Initial/boundary conditions

For a regular solution, we require

$$\tilde{\rho}(0) > 0, \quad \omega(0) = \frac{4-3\gamma}{3},$$
$$\tilde{\rho}(y) \sim y^{-\frac{2}{2-\gamma}} \text{ as } y \to \infty, \quad \lim_{y\to\infty} \omega(y) = 2 - \gamma.$$

NB: this forces the existence of a point where $\gamma\rho^{\gamma-1} - y^2\omega^2 = 0$!

Overview of key difficulties

Sonic point

Let $(\tilde{\rho}(\cdot), \omega(\cdot))$ be a C^1-solution to the self-similar Euler-Poisson system on the interval $(0, \infty)$. A point $y_* \in (0, \infty)$ such that

$$\gamma\tilde{\rho}^{\gamma-1}(y_*) - y_*^2\omega^2(y_*) = 0$$

is called a *sonic point*.

Regularity

Expect stability tied to regularity (MRRS ’22). Requires smoothness through sonic point.

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Non-linear, non-autonomous system

No general recipe for solving such problems. No fixed phase portrait analysis for invariant regions.

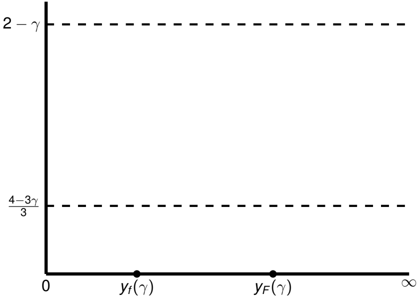
Overview of Strategy

Two explicit solutions

Far-field solution (ρ_f, ω_f) and Friedman solution (ρ_F, ω_F):

$$(\rho_f(y), \omega_f(y)) = (k_\gamma y^{-\frac{2}{2-\gamma}}, 2-\gamma), \qquad (\rho_F(y), \omega_F(y)) = (\frac{1}{6\pi}, \frac{4}{3}-\gamma).$$

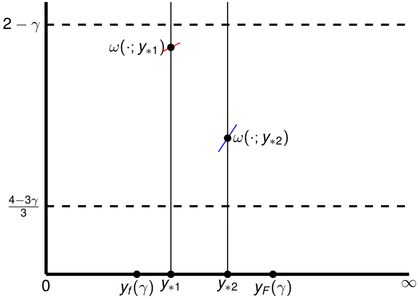
Sonic points at $y_f(\gamma) < y_F(\gamma)$.



Overview of Strategy

Proposition (Local Solution)

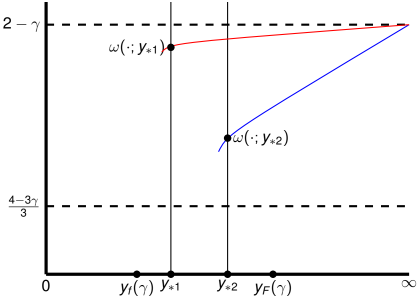
For all $\gamma \in (1, \frac{4}{3})$*, there exists* $\nu > 0$ *such that for all* $y_* \in [y_f(\gamma), y_F(\gamma)]$*, there exists an analytic solution* $(\rho(\cdot; y_*), \omega(\cdot; y_*))$ *to self-similar Euler-Poisson on* $(y_* - \nu, y_* + \nu)$ *with a single sonic point at* y_**.*



Overview of Strategy

Lemma (Solving to the right)

For each $\gamma \in (1, \frac{4}{3})$, each $y_* \in [y_f(\gamma), y_F(\gamma)]$, the local solution $(\rho(\cdot; y_*), \omega(\cdot; y_*))$ obtained by Taylor expansion extends globally to the right on $[y_*, \infty)$, remains supersonic, and satisfies the asymptotic boundary conditions.



Overview of Strategy

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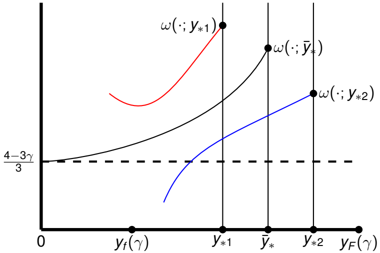
Aim: Find $\bar{y}_*$ such that local solution $(\rho(\cdot;\bar{y}_*), \omega(\cdot;\bar{y}_*))$ extends smoothly to $y = 0$. Look for solution with

$$\frac{4}{3} - \gamma \le \omega(y;\bar{y}_*) < 2 - \gamma, \qquad \lim_{y\to 0} \omega(y;\bar{y}_*) = \frac{4}{3} - \gamma.$$



Stability of Larson–Penston

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Isothermal Euler-Poisson ($\gamma = 1$)

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ &\rho\left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) + \nabla \rho = -\rho \nabla \phi, \\ &\Delta \phi = 4\pi\rho, \quad \lim_{|x|\to\infty} \phi(t,x) = 0, \end{aligned}$$

Larson-Penston self-similar solution:

$$\overline{\rho}(t,r) = \frac{1}{(T-t)^2}\tilde{\rho}\left(\frac{r}{T-t}\right), \quad \overline{u}(t,r) = \tilde{u}\left(\frac{r}{T-t}\right)$$

Theorem (Guo-Hadzic-Jang-S., '24)

The Larson-Penston solution (Euler-Poisson, $\gamma = 1$) is nonlinearly stable against radial perturbations.

Variables for stability

Eulerian

Self-similar variables for stability:

$$s(t) = -\log(T - t), \quad y = r(T - t)^{-1},$$

$$\varrho(t, r) = (T - t)^{-2}\rho(s, y), \quad u(t, r) = v(s, y) - y$$

Perturb density and 'momentum':

$$\rho = \bar{\rho} + \epsilon R, \qquad \rho v = \bar{\rho}\bar{v} + \epsilon P.$$

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Lagrangian

Lagrangian flow map η solves

$$\partial_t \eta(t,r) = u(t,\eta(t,r)),$$

then

$$z = r(T-t)^{-1}, \qquad \zeta(s,z) = (T-t)^{-1}\eta(t,r).$$

For Larson–Penston flow map given as $\overline{\zeta}$, perturbation is

$$\theta = \zeta - \overline{\zeta}.$$

Comparison to existing results

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Euler implosion

Merle–Raphaël–Rodnianski–Szeftel ’22 and subsequent works (Buckmaster–Cao-Labora–Gomez-Serrano–Shi–Staffilani, Chen–Cialdea–Shkoller–Vicol)

- Finite co-dimension stability: accretivity in small backwards cones, finitely many unstable directions (Biasi ’22)
- Non self-adjoint linearised operator, non-explicit coefficients
- Smoothness across sonic line essential
- High-low order energy method

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Semi-linear blowups

Glogic–Donninger–Schörkhuber–Costin (wave maps, hyperbolic Yang-Mills)

- Full stability
- Explicit self-similar solutions
- Fixed sound/null cone

Eulerian vs Lagrangian

Larson-Penston solution

In Lagrangian framework, Larson-Penston flow map $\bar{\zeta}$ satisfies

$$\bar{\zeta}(z) \sim_{z\to 0^+} z^{\frac{1}{3}}, \qquad \bar{\zeta}(z) \sim_{z\to\infty} z.$$

Relates to Eulerian Larson-Penston pseudo-velocity $\overline{\omega}(y) = \overline{u}(y)/y + 1$ by

$$z\partial_z\bar{\zeta}(z) = \overline{\omega}(\bar{\zeta}(z))\bar{\zeta}(z).$$

- Lagrangian: convenient for nonlinear analysis
- Eulerian: convenient for linear analysis

Linear Stability

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Let ϕ be such that $D_y\phi = R$ ($D_y = \partial_y + \frac{2}{y}$ is divergence). In suitable first order formulation, linearised problem is

$$\partial_s \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mathcal{L} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + N \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

- Non-self-adjoint problem (complex eigenvalues);
- Sonic degeneracy and issues with dissipativity (monotonicity);
- Growing mode with $\lambda = 1$ (time-translation).

Linear Stability II

Accretivity

Relatively compact perturbation $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ admits energy-type estimates on interval $z \in [0, Z_0]$ for large Z_0. Yields

$$\left\| e^{\widetilde{\mathcal{L}} s} \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} \right\|_{H_{\text{low}}} \leq C e^{-2\delta s} \left\| \begin{pmatrix} \phi_0 \\ \psi_0 \end{pmatrix} \right\|_{H_{\text{low}}} \qquad (*)$$

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Mode Stability

There exists $\delta > 0$ such that the spectrum of the linearised operator,

$$\sigma_{\mathcal{L}} \setminus \{1\} \subset \{\text{Re}\lambda \leq -\delta\}.$$

Conclusion: After projecting away time-translation mode, $(*)$ holds for $\mathcal{L}$ with δ.

Mode Stability

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Difficulties

- Non-explicit coefficients (depend on LP solution) – elegant argument of Glogic unavailable
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Strategies

- Energy arguments exclude high and low frequency eigenvalues (monotonicity crucial)
- Interval arithmetic to handle intermediate (compact) region

Nonlinear stability

Established in Lagrangian formulation

- High-order energy method with polynomial weights (H_{high})
- Use H_{low} to squeeze some exponential decay to treat bad errors in the interior zone
- Total energy incorporating pointwise control of Lagrangian flow map
- Asymptotic dampening (finite mass and energy) easy via Lagrangian formulation

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Estimates combine with fixed point argument to establish existence of $|T| \ll 1$, such that solution with $s(t) = -\log(T - t)$ exists globally in s-time and decays

Thank you!