The null gluing problem for the Einstein equations

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Gluing problems for the Einstein equations

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$$\mathrm{Ric}(\mathbf{g}) = 0$$

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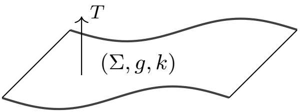
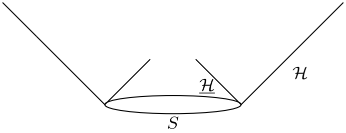
- Einstein equations can be cast as a hyperbolic system of equations admitting well-posed initial value problems.



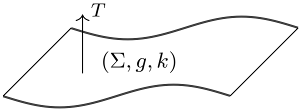
Spacelike initial data.



Characteristic initial data.

Initial value formulations

Initial data cannot be prescribed freely and must satisfy *constraint equations*.



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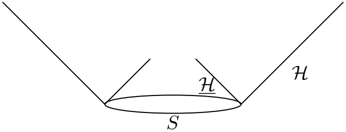

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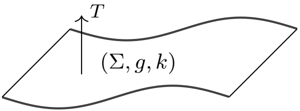

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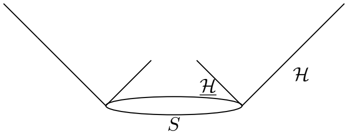

Characteristic initial data.

Elliptic type equations for g and the 2-tensor, k.

$$R(g) = -|k|^2 - (\mathrm{tr} k)^2,$$
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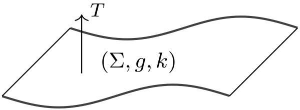

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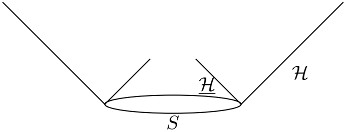

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Transport type equations (*null structure equations*) for metric components, Christoffel symbols, curvature.

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$$\rightsquigarrow$$

Initial data gluing problem: Is it possible to glue two initial data sets of the Einstein equations of the same type to construct a solution of the associated constraint equations?

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J. DIFFERENTIAL GEOMETRY
73 (2006) 185-217

ON THE ASYMPTOTICS FOR THE VACUUM EINSTEIN CONSTRAINT EQUATIONS

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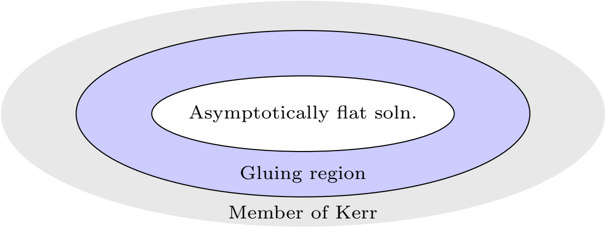
Theorem. (Corvino-Schoen/Chruściel-Delay, 2003) Any asymptotically flat spacelike initial data can be glued to spacelike Kerr black hole initial data with given Kerr parameters $(\mathbf{E}^{\mathrm{Kerr}}, \mathbf{P}^{\mathrm{Kerr}}, \mathbf{L}^{\mathrm{Kerr}}, \mathbf{G}^{\mathrm{Kerr}})$.

The gluing problem for spacelike initial data



Brief outline of the proof:

- Cutoff the asymptotically flat solution to Kerr. The cutoff (gluing) region in general will not solve constraint equations.

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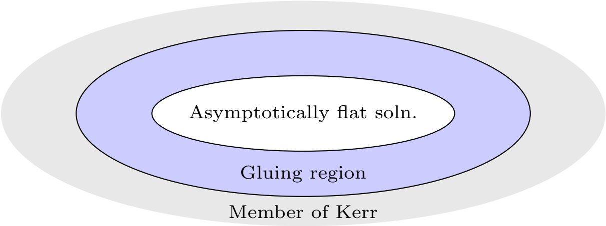

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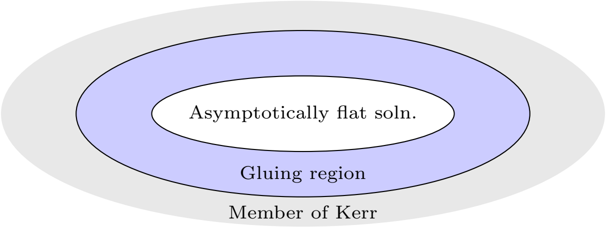

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- Correct the error by solving the **linearised** constraint equations and applying the implicit function theorem.

But, the linearised constraint equations are surjective only up to a 10-dimensional space of obstructions.

$\rightsquigarrow$ The 10 Kerr parameters are used as additional parameters of the problem.

The null gluing problem

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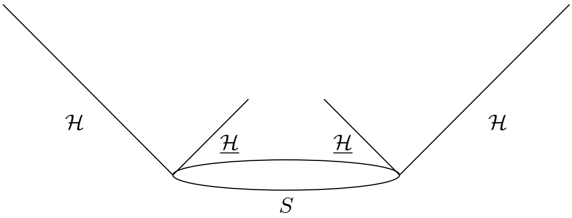
The Characteristic Gluing Problem for the Einstein Vacuum Equations: Linear and Nonlinear Analysis

Stefanos Aretakis, Stefan Czimek and Igor Rodnianski

Characteristic initial data



- Transversal $\underbrace{\text{null hypersurfaces}}_{\text{The metric } \mathbf{g} \text{ is degenerate}}$ $\mathcal{H}$ and $\underline{\mathcal{H}}$ intersecting at spheres, S.
- The null structure equations are transport equations along $\mathcal{H}$ and $\underline{\mathcal{H}}$

The gluing problem for characteristic initial data

Given characteristic initial data on $(\mathcal{H}_1, \underline{\mathcal{H}}_1)$ and $(\mathcal{H}_2, \underline{\mathcal{H}}_2)$, can we glue $\mathcal{H}_1$ to $\mathcal{H}_2$ along a null hypersurface $\mathcal{H}_{\mathcal{G}}$ such that there exists a solution to the null structure equations on $\mathcal{H}_1 \cup \mathcal{H}_{\mathcal{G}} \cup \mathcal{H}_2$?

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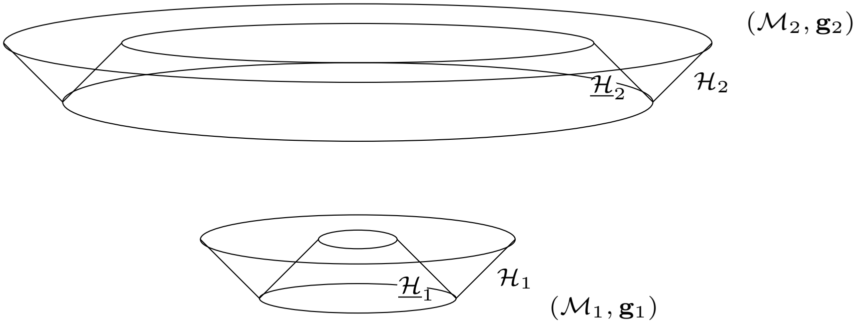

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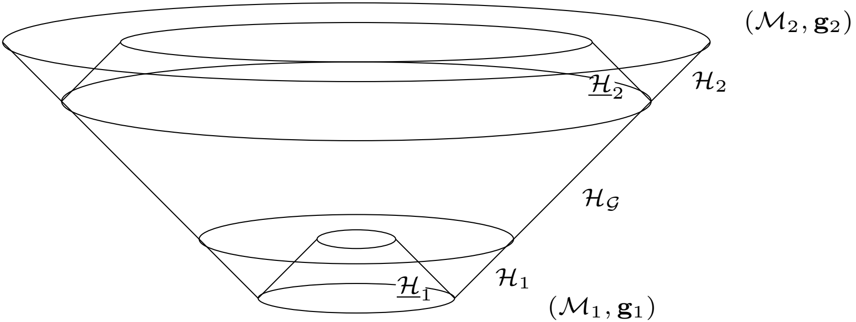

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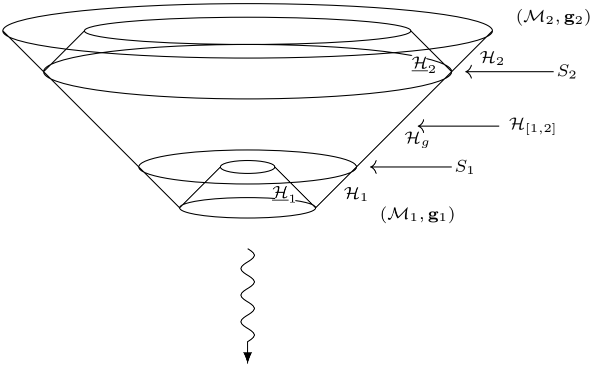
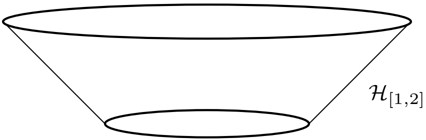
Notation:

- On a sphere $S_1 \subset (\mathcal{M}_1, \mathbf{g}_1)$, sphere data will be denoted by x_1.
- A solution to the null structure equations will be denoted by x.
  It is the tuple of metric components and their derivatives up to order k that satisfy the null structure equations.

The C^k-null gluing problem



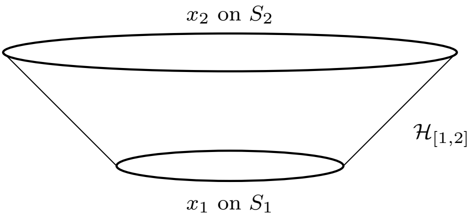


The C^k-null gluing problem

- Let S_1 be a sphere in a spacetime $(\mathcal{M}_1, \mathbf{g}_1)$ and S_2 a sphere in a spacetime $(\mathcal{M}_2, \mathbf{g}_2)$.
- Let x_1 and x_2 be respective sphere data.

Does there exist a null hypersurface $\mathcal{H}_{[1,2]} := \bigcup_{1 \leq v \leq 2} S_v$ and a solution x to the null structure equations on $\mathcal{H}_{[1,2]}$ such that

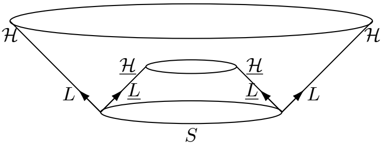
$$x|_{S_1} = x_1, \qquad x|_{S_2} = x_2?$$



Analogous to the spacelike gluing problem, there are **obstructions** to solving this problem.

Double null gauge

To discuss obstructions, we choose to work in *double null gauge*.



Characteristic initial data with transversal intersecting null hypersurfaces $\mathcal{H}$ and $\underline{\mathcal{H}}$ intersecting at spheres S. Generators L and $\underline{L}$.

Work in *double null coordinates* $(u, v, \theta^1, \theta^2)$. Metric has the form

$$\mathbf{g} = -4\Omega^2 du dv + \not{g}_{AB} d\theta^A d\theta^B,$$

$A, B, ... = 1, 2$. $L = \partial_v$, $\underline{L} = \partial_u$.

Null structure equations in double null gauge

Metric in double null coordinates: $\mathbf{g} = -4\Omega^2 du dv + \not{g}_{AB} d\theta^A d\theta^B$, $A, B = 1, 2$.

Christoffel symbols encoded by **Ricci coefficients**:

$$\chi_{AB}, \underline{\chi}_{AB}, \eta_A, \ldots$$

Split: $\chi = \widehat{\chi} + \frac{1}{2}\mathrm{tr}\chi$ with respect to $\not{g}$.

- $\mathrm{tr}\chi$ is **null expansion** along $\mathcal{H}$.
- $\widehat{\chi}$ is **shear** along $\mathcal{H}$.
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Null structure equations:
First variation equation:

$$D\not{g} = 2\Omega\chi, \quad \underline{D}\not{g} = 2\Omega\underline{\chi},$$

$D := \mathcal{L}_L$, $\underline{D} := \mathcal{L}_{\underline{L}}$,
$L = \partial_v$, $\underline{L} = \partial_u$.

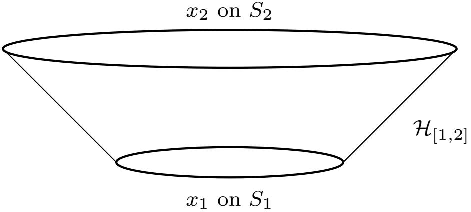
Raychaudhuri equation:

$$\partial_v(\Omega\mathrm{tr}\chi) = -\frac{(\Omega\mathrm{tr}\chi)^2}{2} - \Omega^2|\widehat{\chi}|^2_{\not{g}},$$

$$\vdots$$

More equations depending on the order, k.

Obstructions to solving the C^k-null gluing problem



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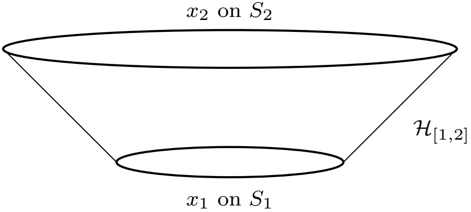

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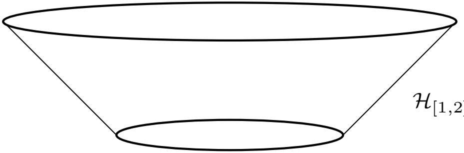

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- Infinite-dimensional space of conservation laws to the linearised C^k-null gluing problem at Minkowski.

  $\rightsquigarrow$ Need to **relax** its formulation

Two types of gauge perturbations

Sphere perturbations: small changes of the sphere data x_2 on S_2.

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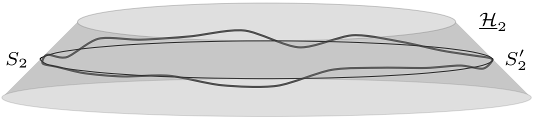
- **Transversal perturbations**: perturb S_2 to a nearby sphere S_2' along an ingoing null hypersurface $\underline{\mathcal{H}}_2$ containing both spheres.
  - The C^k-sphere data x_2' on S_2' can be expressed in terms of x_2, a perturbation function f and the geometry of $\underline{\mathcal{H}}_2$.
  - The perturbation function f is defined through

$$u' = u + f(u, \theta^1, \theta^2).$$



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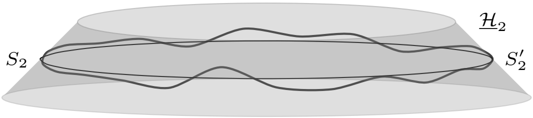

- **Angular perturbations**: sphere diffeomorphisms of sphere data defined by pulling back the sphere data x_2 under a diffeomorphism of the sphere S_2.

The perturbative C^k-null gluing problem

- Let S_1 be a sphere in a spacetime $(\mathcal{M}_1, \mathbf{g}_1)$ and $S_2 \subset \underline{\mathcal{H}}_2$ a sphere in a spacetime $(\mathcal{M}_2, \mathbf{g}_2)$.
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- Let x_1 and x_2 be respective sphere data.

Does there exist:

1. A sphere $S_2' \subset \underline{\mathcal{H}}_2$, arising from a sphere perturbation of S_2, with sphere data x_2' such that there exists a null hypersurface $\mathcal{H}_{[1,2]}$ connecting S_1 to S_2'?
2. A solution x to the null structure equations on $\mathcal{H}_{[1,2]}$ such that

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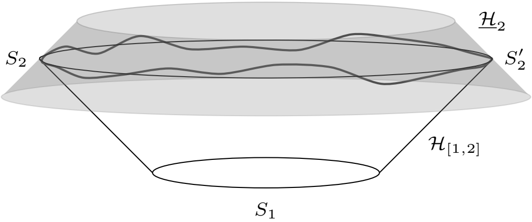

Solvability of the perturbative C^k-null gluing problem

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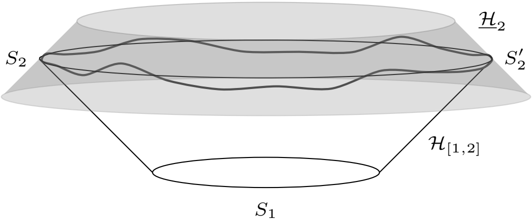

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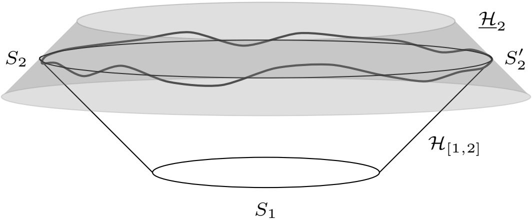

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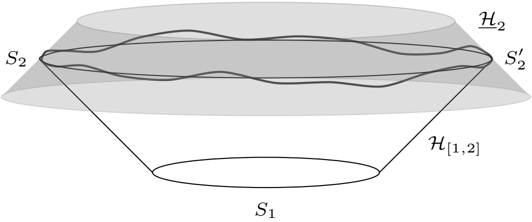

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- An infinite number of **gauge-dependent** obstructions that depend on the sphere perturbation.
- A finite number of **gauge-invariant** obstructions that are linearly invariant under the sphere perturbation.

Solve perturbative C^k-null gluing problem up to the gauge-invariant obstructions.

Approach to solving the perturbative C^k-null gluing problem

The approach to solving perturbative C^k-null gluing problems

1. Linearise the null structure equations at Minkowski.
2. Analyse the linear null structure equations to construct conservation laws.
3. Investigate the gauge dependence of the conservation laws.
4. Construct a solution to the linearised null structure equations up to the gauge-independent conserved quantities.
5. Apply the implicit function theorem to solve the nonlinear null gluing problem close to Minkowski up to the gauge-independent conserved quantities.

Characteristic seed and hierarchy of null structure equations

Redundancy in null structure equations means that not all Ricci coefficients (Christoffel symbols) and null curvature components need to be specified.

Specify a *characteristic seed*:

1. On S_1, ϕ, $\mathrm{tr}\chi$, $\mathrm{tr}\underline{\chi}$, $\widehat{\chi}$, $\widehat{\underline{\chi}}$, η,...
2. On $\mathcal{H}_{[1,2]}$, Ω and $\mathrm{conf}\not{g}$.

$\mathrm{conf}\not{g}$ is the conformal class of metrics on each $S \subset \mathcal{H}$:

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- ⇝ Sphere data need not include the prescription of all Ricci coefficients and null curvature components.
- ⇝ Null structure equations can be solved hierarchically from the characteristic seed.

Exemplifying the approach - 1. Linearisation

Linearisation procedure: Expansion of sphere data

$$x = x^{Minkowski} + \varepsilon \dot{x} + \mathcal{O}(\varepsilon^2).$$

Consider the subset of linearised null structure equations:

$$D\left(\frac{\dot{\phi}}{v}\right) = \frac{(\Omega \dot{\mathrm{tr}} \chi)}{2}, \qquad D\left(D\dot{\phi}\right) = 2D\dot{\Omega},$$

$$D(\Omega \dot{\mathrm{tr}} \chi) = -\frac{2(\Omega \dot{\mathrm{tr}} \chi)}{v},$$

$$D\left(v^2 \dot{\eta} + \frac{v^3}{2} \not{d}\left((\Omega \dot{\mathrm{tr}} \chi) - \frac{4}{v}\dot{\Omega}\right)\right) = \not{\mathrm{div}}_{\mathbb{S}^2} \dot{\hat{\chi}},$$

$D := \mathcal{L}_L$ – derivative along $\mathcal{H}$.

$\dot{\Omega}, \dot{\phi}, (\Omega \dot{\mathrm{tr}} \chi), \dot{\eta}, \dot{\hat{\chi}} \in \dot{x}$.

Exemplifying the approach - 2. Analysis

Combine null structure equations.

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$$\Longrightarrow \qquad D\left(\frac{v}{2}\left((\Omega \dot{\mathrm{tr}}\chi) - \frac{4}{v}\dot{\Omega}\right) + \frac{\dot{\phi}}{v}\right) = 0,$$

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Conserved charge along $\mathcal{H}_{[1,2]}$:

$$\mathcal{Q}_1 := \frac{v}{2}\left((\Omega\dot{\mathrm{tr}}\chi) - \frac{4}{v}\dot{\Omega}\right) + \frac{\dot{\phi}}{v}$$

Exemplifying the approach - 2. Analysis

Null structure equation for $\dot{\eta}$:

$$D\left(v^2\dot{\eta}+\frac{v^3}{2}\not{d}\left((\Omega\dot{\mathrm{tr}}\chi)-\frac{4}{v}\dot{\Omega}\right)\right)=\not{\mathrm{div}}_{\mathbb{S}^2}\dot{\widehat{\chi}}.$$

- The operator $\not{\mathrm{div}}_{\mathbb{S}^2}$ maps 2-tensors on $\mathbb{S}^2$ to vectorfields on $\mathbb{S}^2$.

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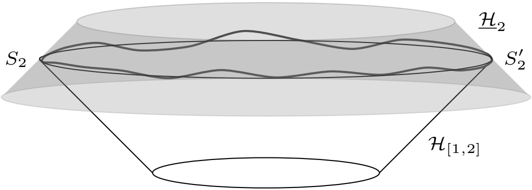
Projecting the above equation onto spherical harmonics with mode $l = 1$ gives the conserved charge along $\mathcal{H}_{[1,2]}$:

$$\mathcal{Q}_2 := v^2\dot{\eta}^{[1]} + \frac{v^3}{2}\not{d}\left((\Omega\dot{\mathrm{tr}}\chi)^{[1]} - \frac{4}{v}\dot{\Omega}^{[1]}\right).$$

Exemplifying the approach - 3. Gauge dependence



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Use **linearised** perturbation function $\dot{f}$ to transform from $S_2 \to S_2'$.

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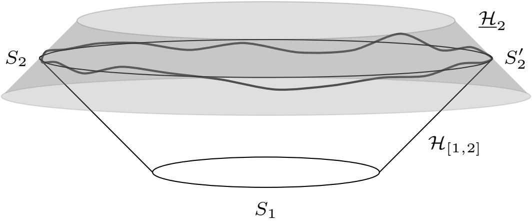

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Use **linearised** perturbation function $\dot{f}$ to transform from $S_2 \to S_2'$.

$$\mathcal{Q}_1 = \frac{1}{2}\left(\mathring{\Delta\!\!\!/}\, \dot{f} - \partial_u \dot{f}\right), \qquad \mathcal{Q}_2 = 0$$

$\Longrightarrow$ $\mathcal{Q}_1$ is **gauge-dependent** and $\mathcal{Q}_2$ is **gauge-independent**.

Solve the null structure equations by integrating along $\mathcal{H}_{[1,2]}$.

For example,

$$\left[\dot{\phi}\right]_1^2 = 2\int_1^2 \dot{\Omega}dv' + v\dot{\phi}(1) + \frac{1}{2}\left((\Omega\dot{\mathrm{tr}}\chi)(1) - 4\dot{\Omega}(1)\right),$$

$$\left[\dot{\not{g}}_c\right]_1^2 = 2\int_1^2 \frac{1}{v'^2}\dot{\hat{\chi}}dv'$$

$$\left[v'^2\dot{\eta} + \frac{v'^3}{2}\not{d}\left((\Omega\dot{\mathrm{tr}}\chi) - \frac{4}{v'}\dot{\Omega}\right)\right]_1^2 = \mathring{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}\left(\int_1^2 \dot{\hat{\chi}}dv'\right) \text{ for modes } l \geq 2.$$

$$\vdots$$

Exemplifying the approach - 4. Solving null structure equations

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- Choose **linear** characteristic seed $\dot{\Omega}$ and $\dot{\hat{\chi}}$ on $\mathcal{H}_{[1,2]}$ such that these equations are satisfied.

Exemplifying the approach - 4. Solving null structure equations

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Repeat steps $1 - 4$ for the rest of the hierarchy of null structure equations.

$\rightsquigarrow$ Gluing for the linear gluing problem up to the conserved charges.

Obstruction spaces for k=2 and k=3

The C^2-null gluing problem of Aretakis, Czimek and Rodnianski

C^2-sphere data contains the information necessary to derive the metric components and their derivatives up to order 2 on a sphere.

The perturbative C^2-null gluing problem:

- Let S_1 be a sphere in a spacetime $(\mathcal{M}_1, \mathbf{g}_1)$ and $S_2 \subset \underline{\tilde{\mathcal{H}}}_2$ a sphere in a spacetime $(\mathcal{M}_2, \mathbf{g}_2)$.
- Let x_1 and x_2 be respective C^2-sphere data.

Does there exist:

1. A sphere $S_2' \subset \underline{\mathcal{H}}_2$, arising from a sphere perturbation of S_2, with C^2-sphere data x_2 such that there exists a null hypersurface $\mathcal{H}_{[1,2]}$ connecting S_1 and S_2'?
2. A solution x to the null structure equations on $\mathcal{H}_{[1,2]}$ such that

$$x|_{S_1} = x_1, \qquad x|_{S_2'} = x_2'?$$

The C^2-null gluing problem of Aretakis, Czimek and Rodnianski

The perturbative C^2-null gluing problem can be solved up to the 10-dimensional space of obstructions $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$.

- Geometric interpretation: The charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ have geometric significance. They correspond to the ADM energy, linear momentum, angular momentum and centre-of-mass, respectively.

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- Geometric interpretation: The charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G})$ have geometric significance. They correspond to the ADM energy, linear momentum, angular momentum and centre-of-mass, respectively.
- The linearisations of $(\dot{\mathbf{E}}, \dot{\mathbf{P}}, \dot{\mathbf{L}}, \dot{\mathbf{G}})$ at Minkowski correspond to gauge-invariant conserved charges in the linearised C^2-null gluing problem along $\mathcal{H}_{[1,2]}$ which are not glued by the methods here.

$$\mathcal{Q}_2 \longleftrightarrow \dot{\mathbf{L}},\ \dot{\mathbf{G}}$$

The C^3-null gluing problem

To solve the C^3-null gluing problem use the solution of the C^2-null gluing problem and:

1. Additional components of sphere data that give 3 derivatives of the metric components. This includes the quantities $D\alpha$ and $\underline{D\alpha}$. $D := \mathcal{L}_L$ and $\underline{D} := \mathcal{L}_{\underline{L}}$.

2. Additional null structure equations corresponding to the additional components of sphere data.

3. Derive charges from the novel transport equations.

4. Ensure the C^2-part "fits" into the C^3-null gluing problem.

The C^3-null gluing problem - Novel obstructions

The novel linear null structure equations for the C^3-sphere data quantity $\underline{D\alpha}$ lead to the following gauge-invariant conserved charges:

$$\mathbf{U}^m := \left(\frac{1}{\phi} \underline{\widehat{D}\alpha} + \frac{1}{\phi^2} (1 + \phi^2 \nabla\!\!\!/ \,\widehat{\otimes}\, \mathrm{div}\!\!\!/ \,) \underline{\alpha} \right)^{(2m)}_{\psi}$$
$$\mathbf{V}^m := \left(\frac{1}{\phi} \underline{\widehat{D}\alpha} + \frac{1}{\phi^2} (1 + \phi^2 \nabla\!\!\!/ \,\widehat{\otimes}\, \mathrm{div}\!\!\!/ \,) \underline{\alpha} \right)^{(2m)}_{\phi}$$

- An additional 10-dimensional space of obstructions.

  $\implies$ There are $10 + 10 = 20$ obstructions to the C^3-null gluing problem.

- The linearisations $\dot{\mathbf{U}}$, $\dot{\mathbf{V}}$ correspond to conserved charges in the linear null structure equations - linearised Bianchi equation for $\underline{D\dot{\alpha}}$.

Codimension-20 perturbative C^3-null gluing

Theorem (S., 2024)

Let x_1 and x_2 denote C^3-sphere data on spheres S_1 and S_2 close to Minkowski sphere data. Moreover, Let S_2 be contained in an ingoing null hypersurface $\underline{\mathcal{H}}_2$. Then there exists

- a solution to the null structure equations x along an outgoing null hypersurface $\mathcal{H}_{[1,2]}$,
- sphere data x_2' on a sphere S_2' arising from a sphere perturbation of $S_2 \in \underline{\mathcal{H}}_2$

such that on S_1 we have that

$$x|_{S_1} = x_1.$$

Moreover, if the gauge-invariant charges $(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}, \mathbf{U}, \mathbf{V})$ match on S_2', that is,

$$(\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}, \mathbf{U}, \mathbf{V})(x|_{S_2'}) = (\mathbf{E}, \mathbf{P}, \mathbf{L}, \mathbf{G}, \mathbf{U}, \mathbf{V})(x_2')$$

then on S_2' it holds that

$$x|_{S_2'} = x_2'.$$

Getting around the gauge-invariant obstructions

1. Linearising around Schwarzschild (instead of Minkowski) with $M > 0$ yields no obstructions.
    - The 20 obstructions are no longer conserved or gauge-invariant.
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2. Consider the compact 2-dimensional intersection of null hypersurfaces to be of higher genus (Chruściel, Cong, Gray).
   - E.g. for the torus, the C^2-null gluing problem has only 7 (instead of 10) gauge-invariant conserved charges.
   - Also showed that the number of charges depends on the spacetime dimension, cosmological constant, and number of higher transverse derivatives.

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3. Obstruction free null gluing of Czimek-Rodnianski overcomes all obstructions to the C^2-null gluing problem by using the structure of the **quadratic** terms in the null structure equations.

Thank you! Questions?