Spacelike initial data for black hole stability

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Cauchy formulation of general relativity

- **Cauchy data:**
  - (Σ, g) a Riemannian $3-$manifold,
  - π a symmetric 2-tensor on Σ.
- **Cauchy formulation of EVE:** find $(\mathcal{M}, \mathbf{g})$ with $\mathbf{Ricci}(\mathbf{g}) = 0$ and
  - Σ is a spacelike hypersurface in $\mathcal{M}$,
  - (g, π) are the first and (reduced) second fundamental form $\pi = k - (\mathrm{tr}_g k) g$.

Theorem (Choquet-Bruhat 1952, Choquet-Bruhat–Geroch 1969)
Cauchy problem locally solvable $\iff$ (Σ, g, π) *solve constraint*

$$\begin{cases} R(g) + \frac{1}{2}(\mathrm{tr}_g \pi)^2 - |\pi|_g^2 = 0, \\ \mathrm{div}_g \pi = 0. \end{cases} \iff \Phi(g, \pi) = 0 \qquad \text{(C)}$$

Stability of Kerr

Theorem

Let $|a| < m$. Consider data on Σ of the form

$$(g, \pi) = \left(g_{m,a} + O\left(\varepsilon r^{-q-\delta}\right), \pi_{m,a} + O\left(\varepsilon r^{-q-\delta-1}\right)\right).$$

*Stability in **region** II holds with*

- (Caciotta–Nicolò 2010) $q \geq 3$ *and* $0 < \delta < 1$.
- (Shen 2023) $q \geq 1$ *and* $0 < \delta < 1$.

*Stability in **region** III holds with*

- (Klainerman–Szeftel 2018) $a = 0$, $q = 1$ *and* $\delta = \left(\frac{1}{2}\right)_+$.
- (D–H–R–T 2021) $a = 0$, $q = 2$ *and* $\delta = \left(\frac{1}{2}\right)_+$.
- (K–S + G–K–S + S 2021) $|a| \ll m$, $q = 1$ *and* $\delta = \left(\frac{1}{2}\right)_+$.

A brief review of the constraint literature

Conformal method(s):

- reduces the constraint equations to elliptic system for (φ, W)

$$\begin{cases} 8\Delta_\gamma\varphi = R(\gamma)\varphi + \frac{2}{3}\tau^2\varphi^5 - |\sigma + K_\gamma W|^2_\gamma \varphi^{-7}, \\ \Delta_{\gamma,conf}W = \frac{2}{3}\varphi^6 \mathrm{d}\tau. \end{cases}$$

- get solutions with weak decay ($q = 0$) but far from CMC.

Gluing construction:

- glue any inner AF solution to exact Kerr,
- linear obstructions due to Minkowski symmetries.

General perturbative recipe I

1. Start from an AF exact solution $(\Sigma, \bar{g}, \bar{\pi})$.

2. Add a decaying perturbation $|\check{g}| + r|\check{\pi}| \lesssim \varepsilon r^{-q-\delta}$:

$$g = \bar{g} + \check{g} \qquad \pi = \bar{\pi} + \check{\pi}$$

3. Correct it with Corvino-Schoen conformal method:

$$g = (1+u)^4(\bar{g} + \check{g}), \qquad \pi = \bar{\pi} + \check{\pi} + L_{\bar{g}}X.$$

For such (g, π) the constraint equations $\Phi(g, \pi) = 0$ become

$$P(u, X) = D\Phi[\bar{g}, \bar{\pi}](\check{g}, \check{\pi}) + NL$$

with the required decay

$$|u| + |X| \lesssim \varepsilon r^{-q-\delta}$$

General perturbative recipe II

$$P(u, X) = D\Phi[\bar{g}, \bar{\pi}](\check{g}, \check{\pi}) + NL, \qquad |u| + |X| \lesssim \varepsilon r^{-q-\delta}$$

4. Consider $P : H^2_{-q-\delta}(\Sigma) \longrightarrow L^2_{-q-\delta-2}(\Sigma)$.

$$D\Phi[\bar{g}, \bar{\pi}](\check{g}, \check{\pi}) + NL \in \mathrm{im}(P) \iff D\Phi[\bar{g}, \bar{\pi}](\check{g}, \check{\pi}) + NL \perp \ker(P^*)$$

Modify the ansatz with compactly supported $(\breve{g}, \breve{\pi})$:

$$g = (1+u)^4(\bar{g} + \check{g} + \breve{g}), \qquad \pi = \bar{\pi} + \check{\pi} + \breve{\pi} + L_{\bar{g}} X.$$

5. Orthogonality conditions to ensure solvability become

$$\Big\langle (\breve{g}, \breve{\pi}), D\Phi[\bar{g}, \bar{\pi}]^*(W) \Big\rangle = -\Big\langle D\Phi[\bar{g}, \bar{\pi}](\check{g}, \check{\pi}) + NL, W \Big\rangle, \quad \forall\, W \in \ker(P^*)$$

Question: what is $\ker(D\Phi[\bar{g}, \bar{\pi}]^*) \cap \ker(P^*)$?

Cultural break

Theorem (McOwen 1979)

Consider $q \in \mathbb{N}$, $0 < \delta < 1$ *and* $\Delta : H^2_{-q-\delta}(\mathbb{R}^3) \longrightarrow L^2_{-q-\delta-2}(\mathbb{R}^3)$.

- *If* $q = 0$, Δ *is an isomorphism.*
- *If* $q \geq 1$, Δ *is injective with*

$$\ker(\Delta^*) = \Big\{ \textit{harmonic polynomials of degree} \leq q-1 \Big\}.$$

Theorem (Moncrief 1975)

If $(\Sigma, \bar{g}, \bar{\pi})$ *gives rise to* $(\mathcal{M}, \mathbf{g})$ *solving the Einstein vacuum equations, then*

$$\ker\left(D\Phi[\bar{g}, \bar{\pi}]^*\right) \simeq \Big\{ \textit{Killing vector fields of } (\mathcal{M}, \mathbf{g}) \Big\}.$$

Elements of $\ker\left(D\Phi[\bar{g}, \bar{\pi}]^*\right)$ *are called the KIDS.*

General perturbative recipe III

Question: what is $\ker(D\Phi[\bar{g},\bar{\pi}]^*) \cap \ker(P^*)$?

Bad news:

$$P(u,X) = D\Phi[\bar{g},\bar{\pi}](4u\bar{g}, 2u\bar{\pi} + L_{\bar{g}}X) \Longrightarrow \ker(D\Phi[\bar{g},\bar{\pi}]^*) \subset \ker(P^*)...$$

...but the boundary condition for the elliptic system is not prescribed by the constraint equations!

For $\Sigma \simeq \mathbb{R}^3 \setminus \{r < 1\}$, consider the elliptic operator with Robin boundary condition

$$P_F := \Big((P, B+F) : H^2_{-q-\delta}(\Sigma) \longrightarrow L^2_{-q-\delta-2}(\Sigma) \times H^{\frac{1}{2}}(\partial\Sigma)\Big)$$

where $P = \Delta_{\bar{g}} + \text{l.o.t}$ and $B = \partial_\nu + \text{l.o.t.}$

Choice of F

$$P_F = \Big((P, B + F) : H^2_{-q-\delta}(\Sigma) \longrightarrow L^2_{-q-\delta-2}(\Sigma) \times H^{\frac{1}{2}}(\partial\Sigma) \Big)$$

Proposition

There exists $F \in C^\infty(\partial\Sigma)$ such that

1. *if $q \geq 0$ then P_F is injective,*
2. *$F_{|\mathcal{U}} = 0$ and $F_{|\mathcal{V}} = 1$ for $\mathcal{U}, \mathcal{V} \subset \partial\Sigma$ two open subsets.*

- First part $\longrightarrow$ characterize $\ker(P_F^*)$.
- Second part $\longrightarrow$ distinguish $\ker(P_F^*)$ and $\ker(D\Phi[\bar{g}, \bar{\pi}]^*)$.

Choice of F

$$P_F = \Big((P, B + F) : H^2_{-q-\delta}(\Sigma) \longrightarrow L^2_{-q-\delta-2}(\Sigma) \times H^{\frac{1}{2}}(\partial\Sigma)\Big)$$

Proposition

There exists $F \in C^\infty(\partial\Sigma)$ such that

1. *if $q \geq 0$ then P_F is injective,*
2. *$F_{|\mathcal{U}} = 0$ and $F_{|\mathcal{V}} = 1$ for $\mathcal{U}, \mathcal{V} \subset \partial\Sigma$ two open subsets.*

Let F_0 such that $\dim(\ker(P_{F_0})) > 0$ and $\mathcal{W} \subset \partial\Sigma$ open and not dense. Using (Bartnik 1986) there exists F_1 such that

$$\dim(\ker(P_{F_1})) < \dim(\ker(P_{F_0})) \quad \text{and} \quad F_{1|\mathcal{W}} = F_{0|\mathcal{W}}.$$

Start from F_0 satisfying $F_{0|\mathcal{U}} = 0$ and $F_{0|\mathcal{V}} = 1$ and iterate with $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$.

Characterization of $\ker(P_F^*)$

Corollary

We have $\dim\left(\ker(P_F^*)\right) = 4q^2$ *and*

$$(\mathbb{Y}, 0) \in \operatorname{im}(P_F) \iff \mathbb{Y} \perp \ker(P_F^*).$$

Consider

$$\mathbb{W}_{j,\ell}^{(e)} = \text{harmonic polynomials of degree } j-1 \text{ for the Euclidean metric}$$

and solve

$$P_F^*\left(\widetilde{\mathbb{W}}_{j,\ell}\right) = -P_F^*\left(\mathbb{W}_{j,\ell}^{(e)}\right) \in L^2_{j-4}$$

Injectivity for $P_F \implies$ surjectivity for P_F^*. Then

$$\left\{\mathbb{W}_{j,\ell}^{(e)} + \widetilde{\mathbb{W}}_{j,\ell} \;\middle|\; 1 \le j \le q, -(j-1) \le \ell \le j-1\right\}$$

is a basis of $\dim\left(\ker(P_F^*)\right)$.

The KIDS are alright

Lemma

If $\mathbb{W} \in \ker(P_F^*)$, *then*

$$D\Phi[\bar{g},\bar{\pi}]^*(\mathbb{W}) = 0 \text{ } on \text{ } \{r \in I\} \Longrightarrow \mathbb{W} = 0 \text{ } on \text{ } \Sigma.$$

1. Analyticity on a NBH of $\partial\Sigma \Longrightarrow D\Phi[\bar{g},\bar{\pi}]^*(\mathbb{W}) = 0$ on a NBH of $\partial\Sigma$.
2. Overdetermination of the KIDS $\Longrightarrow \mathbb{W}$ and $B^*(\mathbb{W})$ analytic on $\partial\Sigma$.
3. By construction $F_{|\mathcal{U}} = 0$ and $F_{|\mathcal{V}} = 1$.
   - $(B^* + F^*)(\mathbb{W}) = 0 \Longrightarrow B^*(\mathbb{W})_{|\mathcal{U}} = 0 \Longrightarrow B^*(\mathbb{W}) = 0$ on $\partial\Sigma$
   - $F^*\mathbb{W} = 0 \Longrightarrow \mathbb{W}_{|\mathcal{V}} = 0 \Longrightarrow \mathbb{W} = 0$ on $\partial\Sigma$
4. Thanks to unique continuation:

$$\begin{cases} P^*(\mathbb{W}) = 0 & \text{on } \Sigma \\ B^*(\mathbb{W}) = \mathbb{W} = 0 & \text{on } \partial\Sigma \end{cases} \Longrightarrow \mathbb{W} = 0 \quad \text{on } \Sigma$$

Solving the constraint

- We want to solve

$$P_F(u,X) = \Big(D\Phi[\bar g,\bar\pi]\big((\check g,\check\pi)+(\breve g,\breve\pi)\big) + NL\big(u,X,\breve g,\breve\pi\big), 0 \Big)$$

- Solvable $\iff$ orthogonality conditions against $\ker(P_F^*) = \mathrm{span}\big(\mathbb{W}_{j,\ell}\big)$

$$\Big\langle (\breve g,\breve\pi), D\Phi[\bar g,\bar\pi]^*(\mathbb{W}_{j,\ell}) \Big\rangle = -\Big\langle D\Phi[\bar g,\bar\pi](\check g,\check\pi) + NL\big(u,X,\breve g,\breve\pi\big), \mathbb{W}_{j,\ell} \Big\rangle$$

- Consider

$$(\breve g,\breve\pi) \in \mathrm{span}\Big(\chi(r) D\Phi[\bar g,\bar\pi]^*(\mathbb{W}_{j,\ell})\Big)$$

where χ is supported in the analyticity region away from the boundary.

- Orthogonality conditions are solvable: the matrix

$$\Big(\Big\langle \chi(r) D\Phi[\bar g,\bar\pi]^*(\mathbb{W}_{j,\ell}), D\Phi[\bar g,\bar\pi]^*(\mathbb{W}_{j',\ell'}) \Big\rangle\Big)_{j,j',\ell,\ell'}$$

is invertible thanks to the KIDS Lemma.

The result

Theorem (Fang–Szeftel–T 2024)
Let $(\Sigma, \bar{g}, \bar{\pi})$ *be AF initial data set with* $\Sigma \simeq \mathbb{R}^3 \setminus \{r < 1\}$.
Assume analyticity near and at $\partial\Sigma$.
For all $(\check{g}, \check{\pi})$ *such that*

$$\|\check{g}\|_{H^2_{-q-\delta}(\Sigma)} + \|\check{\pi}\|_{H^1_{-q-\delta-1}(\Sigma)} \leq \varepsilon,$$

there exists a solution of the constraint equations of the form

$$(g, \pi) = \left((1+u)^4 \left(\bar{g} + \check{g} + \breve{g}\right), \bar{\pi} + \check{\pi} + \breve{\pi} + L_{\bar{g}} X\right),$$

with $(\breve{g}, \breve{\pi})$ *compactly supported and in a space of dimension* $4q^2$ *and with*

$$\|(u, X)\|_{H^2_{-q-\delta}(\Sigma)} + \|(\breve{g}, \breve{\pi})\| \lesssim \varepsilon.$$

In particular $|\bar{g} - g| + r|\bar{\pi} - \pi| \lesssim \varepsilon r^{-q-\delta}$.

Conclusive remarks

- **Analyticity of the KIDS:** at leading order

$$D\Phi[\bar{g},\bar{\pi}](h,\varpi) = \big(\Delta_{\bar{g}}\mathrm{tr}_{\bar{g}}h - \mathrm{div}_{\bar{g}}\mathrm{div}_{\bar{g}}h, \mathrm{div}_{\bar{g}}\varpi\big)$$

$$\Downarrow$$

$$D\Phi[\bar{g},\bar{\pi}]^*(f,X) = \big(\mathrm{Hess}_{\bar{g}}f - (\Delta_{\bar{g}}f)\bar{g}, \mathcal{L}_X\bar{g}\big)$$

  Therefore KIDS satisfy an elliptic system on Σ and on $\partial\Sigma$.

- **Application to gluing:** glue to arbitrarily decaying perturbations of Kerr.

- **Application to black holes:** need careful choice of the initial hypersurface entering the horizon.

- **Link with Allen's talk!**