Two remarkable differential operators acting on symmetric 2-tensors

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References

Open acces on HAL= French ArXiv

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2. Le laplacien conforme sur les 2-tenseurs symétriques [hal-04058252v2]
3. Instabilité des métriques de Schwarzschild-Tangherlini riemanniennes [hal-04969211]

Notations

- The Lichnerowicz laplacian

$$\Delta_L h_{ij} = -\nabla^k\nabla_k h_{ij} + R_{ik}h^k_j + R_{jk}h^k_j - 2R_{ikjl}h^{kl},$$

where R_{ij} is the Ricci curvature of g and R_{ijkl} its Riemann curvature.

$$\Delta_L = \nabla^*\nabla + 2(\mathrm{Ric} - \mathrm{Riem}) = \Delta + 2(\mathrm{Ric} - \mathrm{Riem}).$$

- The divergence acting on symmetric 2-tensors

$$(\mathrm{div} h)_j := -\nabla^i h_{ij} = \frac{1}{2}(\mathcal{L}^* h)_j,$$

- The killing operator

$$(\mathcal{L}w)_{ij} = \nabla_i w_j + \nabla_j w_i.$$

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•The exterior differential acting on functions or 1-forms is denoted by d and its L^2 adjoint, the divergence is denoted by d^*.

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$$\Delta_H = dd^* + d^*d = \nabla^*\nabla + \mathrm{Ric} = \Delta + \mathrm{Ric}.$$

• The Lichnerowicz laplacian is also equal to

$$\Delta_L = 2(d_\nabla^* d_\nabla + \mathrm{div}^*\mathrm{div}) - \nabla^*\nabla,$$

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$$C^\infty(M, \mathring{S}_2) = \mathrm{Im}\mathring{\mathcal{L}} \oplus \ker \mathrm{div}$$

$$h = \mathring{\mathcal{L}}w + h_{TT}.$$

Choose h and solve the elliptic equation

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1. Bourguignon, Ebin and Marsden (1976), on closed manifolds (infinite dimensionnal space of smooth TT).
2. Beig (1996) on some conformally flat 3-Manifolds using an operator of order 3.
3. Dain and Friedrich (2001) Smooth on $\mathbb{R}^3$ using spherical harmonics.
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"Let T_0 be a compactly supported traceless symmetric 2-tensor. Define

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then it is easily verified that the tensor T_{ij} defined as

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Theorem (D-2023)

On a smooth Einstein riemannian manifold (M, g) of dimension $n \geq 3$ with $Ric(g) = \lambda g$, the self adjoint operator

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send any trace free symmetric two tensors to a TT-tensor.

$$Tr\,P = 0\,, \quad div\;P = 0, \qquad and \qquad P\,\mathring{\mathcal{L}} = 0.$$

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Applications :

1. Explicit construction of TT-tensors (with compact support if needed).
2. Approximate weakly regular TT-tensors by smooth TT-tensors:
   Example : If $h \in H^1$ is TT and assume that $h = Pu$ with $u \in H^5$ (TT or not). If $u_\epsilon \in C^\infty$ tends to u in H^5 then $h_\epsilon = Pu_\epsilon$ are smooth TT-tensors that tends to h in H^1.
3. Approximate TT-tensors is some weighted spaces by compaclty supported smooth TT-tensors.

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TT-tensors

TT-tensors machine on Einstein manifolds : proof 1

We have $\operatorname{Tr}\Delta_L = \Delta\operatorname{Tr}$, and $\operatorname{Tr}\mathring{\mathcal{L}} = 0$ so $\operatorname{Tr} P = 0$.

Let $c = \frac{n}{n-1}$. We have

$$\operatorname{div}\Delta_L = \Delta_H \operatorname{div},$$

and

$$\operatorname{div}\mathring{\mathcal{L}} = (dd^* + \frac{2}{c}d^*d - 2\lambda),$$

so $\operatorname{div} P$ is equal to

$$\left[(\Delta_H - 2\lambda)(\Delta_H - c\lambda) - (dd^* + \frac{2}{c}d^*d - 2\lambda)(dd^* + \frac{c}{2}d^*d - c\lambda)\right]\operatorname{div}$$

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$$\mathring{\mathrm{Ric}}(g) = \mathrm{Ric}(g) - \frac{1}{n} R(g) g,$$

and the Schouten tensor

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$$\mathring{\mathrm{Ric}}\mathrm{Sch} := \mathring{\mathrm{Ric}} \circ \mathrm{Sch},$$

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$$\mathrm{Sch}(e^{tf}g) = \mathrm{Sch}(g) - \frac{n-2}{2}t\nabla\nabla f + O(t^2)$$

$$\mathrm{Sch}(e^{tf}g) = \frac{(n-2)\lambda}{2(n-1)}\left(g - \frac{(n-1)}{\lambda}t\nabla\nabla f + O(t^2)\right).$$

Let ϕ_t the local flow for $X = -\frac{(n-1)}{\lambda}\nabla f$,

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The Branson operator on 1-forms (1982) :

$$L_B = d^*d + \frac{n-4}{n}dd^* + \frac{n(n-4)}{4(n-2)(n-1)}R - \frac{n-4}{(n-2)}\text{Ric}.$$

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Erdmenger and Osborn (1998), Matsumoto (2013) :

There exist a similar second order operator P_g acting on trace free symmetric two tensors. Moreover if g is Einstein with $\text{Ric}(g) = 2\tilde{\lambda}(n-1)g$ and h is a TT-tensor, then

$$P_g h = \Delta_L h - \left[4\tilde{\lambda}(n-1) - \tilde{\lambda} n \left(\frac{n}{2} - 1\right)\right] h.$$

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Theorem (D-2024)

On a riemannian manifold (M, g) of dimension $n \geq 3$, the self adjoint operator

$$P_g = \Delta_L - \frac{4}{n+2}\mathring{\mathcal{L}}div - 2Ric + \frac{2}{n}\langle Ric(g), .\rangle\, g + \frac{n-2}{4(n-1)}R\,,$$

acting on trace free symmetric two tensors is conformally covariant: $\forall v \in C^\infty(M)$, $\forall u \in C^\infty(M, \mathring{\mathcal{S}}_2)$,

$$P_{e^{2v}g}(u) = e^{-\frac{n-2}{2}v} P_g(e^{\frac{n-6}{2}v}u).$$

- Related result for a Wave type equation on symmetric tensors also by Ben Achour, Huguet and Renaud (2014), Quéva (2015).
- Remark : one can add a weyl curvature term in P_g.

Conformally covariant Laplacian on $\mathring{S}_2$

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An Einstein metric is semi stable if for all TT-tensors "$h \in C_c^\infty(M, \mathring{S}_2)$"

$$\langle \Delta_E h, h \rangle_{L^2(g)} := \int_M \langle \Delta_E h, h \rangle d\mu_g \geq 0,$$

and unstable otherwise.

Theorem (Biquard and Ozuch 2025)

Let (M, g) be an Einstein 4-manifold which is conformal to a Kähler metric. Suppose (M, g) is compact with positive scalar curvature, or (M, g) is one of the known examples of ALF gravitational instantons. Then if (M, g) is not half-conformally flat, then it is unstable.

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The riemannian Schwarzschild-Tangherlini metric, on $\mathbb{R}^2 \times X^{n-2}$ is the Ricci flat metric :

$$g_{Sch} = \frac{1}{(1 - \frac{2m}{r^{n-3}})} dr^2 + (1 - \frac{2m}{r^{n-3}}) d\theta^2 + r^2 g_X$$

where g_X is an Einstein metric with $\mathrm{Ric}(g_X) = (n-3) g_X$. Here $\mathbb{R}^2$ minus the origin correspond to

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Thank you !

PROPAGATEURS ET COMMUTATEURS EN RELATIVITÉ GÉNÉRALE 27

où ε est le tenseur-indicateur de Kronecker. La formule précédente peut être mise sous la forme :

(10.2)
$$(\Delta T)_{\alpha_1\dots\alpha_p} = -\nabla^\rho\nabla_\rho T_{\alpha_1\dots\alpha_p} + \sum_k R_{\alpha_k\mu} T_{\alpha_1\dots}{}^{\mu}{}_{\dots\alpha_p} - \sum_{k\neq l} R_{\alpha_k\rho,\alpha_l\sigma} T_{\alpha_1\dots}{}^{\rho\dots\sigma}{}_{\dots\alpha_p}$$

où dans le deuxième terme du second membre μ occupe la k^e place, dans le troisième terme ρ et σ respectivement les k^e et l^e places.

b) Pour tout tenseur T (antisymétrique ou non), *nous appelons laplacien du tenseur* T *et désignons par* ΔT *le tenseur défini par la formule* (10.2). L'opérateur qui coïncide ainsi sur

André Lichnerowicz (1915-1998)