The conformal method is not conformal

Romain Gicquaud

Institut Denis Poisson
(Université de Tours)

June 12 2025
Journées André Lichnerowicz

Outline

Outline

The Einstein constraint equations

Definition

A vacuum space-time $(\mathcal{M}, h)$ is a Lorentzian manifold (i.e. h has signature $- + \cdots +$) that satisfy some further assumptions (global hyperbolicity) and Einstein's vacuum equations :

$$G^h := \mathrm{Ric}^h - \frac{\mathrm{Scal}^h}{2} h = 0.$$

Definition

Given $M \subset \mathcal{M}$ a (two-sided) spacelike hypersurface, i.e. so that the first fundamental form

$$\widehat{g} := h|_{TM}$$

is positive definite, let ν denote the unit timelike vector $(h(\nu, \nu) = -1)$ orthogonal to TM. We let $\widehat{K}$ be the second fundamental form to M in $\mathcal{M}$:

$$\widehat{K}(X, Y) := h(X, {}^h\nabla_Y \nu).$$

The Einstein constraint equations

As a consequence of the Gauss-Codazzi equations for a hypersurface, we have :

The Einstein constraint equations

As a consequence of the Gauss-Codazzi equations for a hypersurface, we have :

- $0 = 2G^h(\nu, \nu) = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}$ (Hamiltonian constraint),

The Einstein constraint equations

As a consequence of the Gauss-Codazzi equations for a hypersurface, we have :

- $0 = 2G^h(\nu, \nu) = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}$ (Hamiltonian constraint),
- $0 = G^h(\nu, \cdot) = \mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}} \widehat{K})$ (Momentum constraint).

The Einstein constraint equations

As a consequence of the Gauss-Codazzi equations for a hypersurface, we have :

- $0 = 2G^h(\nu, \nu) = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}$ (Hamiltonian constraint),
- $0 = G^h(\nu, \cdot) = \mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}} \widehat{K})$ (Momentum constraint).

These two equations form the **constraint equations** in general relativity.

The Einstein constraint equations

As a consequence of the Gauss-Codazzi equations for a hypersurface, we have :

- $0 = 2G^h(\nu, \nu) = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}$ (Hamiltonian constraint),
- $0 = G^h(\nu, \cdot) = \mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}} \widehat{K})$ (Momentum constraint).

These two equations form the **constraint equations** in general relativity.

Theorem (Y. Choquet-Bruhat – R. Geroch)

Conversely, given a triple $(M, \widehat{g}, \widehat{K})$, we can find a spacetime $(\mathcal{M}, h)$ and an embedding $M \hookrightarrow \mathcal{M}$ such that

- $\widehat{g}$ is the first fundamental form of $M \subset \mathcal{M}$,
- $\widehat{K}$ is the second fundamental form of M.

The Einstein constraint equations

Our goal in this talk is to study a way to construct solutions to the constraint equations :

$$\begin{cases} 0 = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}, \\ 0 = \mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}} \widehat{K}) \end{cases}$$

The Einstein constraint equations

Our goal in this talk is to study a way to construct solutions to the constraint equations :

$$\begin{cases} 0 = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}, \\ 0 = \mathrm{div}_{\widehat{g}} \widehat{K} - d(\mathrm{tr}_{\widehat{g}} \widehat{K}) \end{cases}$$

The method we will study is the **conformal method** due to A. Lichnerowicz, Y. Choquet-Bruhat, J. York...

The Einstein constraint equations

Our goal in this talk is to study a way to construct solutions to the constraint equations :

$$\begin{cases} 0 = \mathrm{Scal}^{\widehat{g}} + (\operatorname{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}, \\ 0 = \operatorname{div}_{\widehat{g}} \widehat{K} - d(\operatorname{tr}_{\widehat{g}} \widehat{K}) \end{cases}$$

The method we will study is the **conformal method** due to A. Lichnerowicz, Y. Choquet-Bruhat, J. York...
In what follows we will make the assumption that M is a compact manifold of dimension $n \geq 3$.

The Einstein constraint equations

Our goal in this talk is to study a way to construct solutions to the constraint equations :

$$\begin{cases} 0 = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}}\, \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}, \\ 0 = \mathrm{div}_{\widehat{g}}\, \widehat{K} - d(\mathrm{tr}_{\widehat{g}}\, \widehat{K}) \end{cases}$$

The method we will study is the **conformal method** due to A. Lichnerowicz, Y. Choquet-Bruhat, J. York...

In what follows we will make the assumption that M is a compact manifold of dimension $n \geq 3$.

The strategy consists in decomposing a given solution $(\widehat{g}, \widehat{K})$ into given data (seed data) and unknowns to transform the constraint equations into an elliptic problem.

Outline

Decomposition for the metric $\widehat{g}$

The most natural choice for $\widehat{g}$ is to look for it in a **conformal class**, i.e. we write

$$\widehat{g} = \varphi^{\kappa} g, \qquad \kappa = \frac{4}{n-2}.$$

with φ unknown. This gives the right amount of degrees of freedom for the Hamiltonian constraint :

$$0 = \mathrm{Scal}^{\widehat{g}} + (\mathrm{tr}_{\widehat{g}} \widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}}.$$

Decomposition for $\widehat{K}$

For $\widehat{K}$, we first split it into its trace part and its traceless part (i.e. into irreducible associated $(\mathrm{C})\mathrm{O}(n, \mathbb{R})$ bundles) :

$$\widehat{K} = \frac{\tau}{n}\widehat{g} + \mathring{K}.$$

This has to do with the fact that the divergence operator has different conformal covariance properties on (sections of) these two bundles (each part is a Stein-Weiss operator) :

Proposition

If T is a symmetric traceless 2-tensor, we have

$$\mathrm{div}_{\widehat{g}}(\varphi^{-2} T) = \varphi^{-2-\kappa} \mathrm{div}_g(T).$$

We wrote

$$\widehat{K} = \frac{\tau}{n}\widehat{g} + \mathring{K}.$$

But this decomposition is not enough to provide an elliptic system because the momentum constraint is a vector equation (actually a 1-form equation) :

$$0 = \mathrm{div}_{\widehat{g}}\, \widehat{K} - d(\mathrm{tr}_{\widehat{g}}\, \widehat{K}).$$

So we need to decompose $\mathring{K}$ further.

York's decomposition

Assume that g has no conformal Killing vector field, i.e. vector fields X such that

$$\mathbb{L}X = 0, \quad \text{where } \mathbb{L}X = \mathring{\mathcal{L}}_X g.$$

There is a L^2-orthogonal decomposition of $\Gamma(\mathrm{S\mathring{y}m}_2(M))$ as follows :

$$\Gamma(\mathrm{S\mathring{y}m}_2(M)) = \mathrm{TT}(M, g) \oplus \mathrm{Im}(\mathbb{L}),$$

where $TT(M)$ is the set of TT-tensors of M (i.e. such that $\mathrm{tr}_g\, \sigma \equiv 0$ and $\mathrm{div}_g(\sigma) \equiv 0$).

Decomposition for $\widehat{K}$

We apply York's decomposition to $\varphi^2 \mathring{K}$ to get

$$\varphi^2 \mathring{K} = \sigma + \mathbb{L} W.$$

Finally, we arrive at

$$\widehat{g} = \varphi^\kappa g, \qquad \widehat{K} = \frac{\tau}{n}\widehat{g} + \varphi^{-2}(\sigma + \mathbb{L} W).$$

And, in agreement with the constraint equations, we choose the following splitting into seed data and unknowns :

- Seed data : g, τ, σ,
- Unknowns : φ, W.

Note that τ, as it is chosen, is the **mean curvature** of the embedding $M \hookrightarrow \mathcal{M}$ into the space-time with initial data $(M, \widehat{g}, \widehat{K})$.

The conformal constraint equations

With this decompostion performed, we can write the constraint equations in terms of the variables we have introduced :

The conformal constraint equations

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\ \varphi = -\frac{n-1}{n}\tau^2\varphi^{\kappa+1} + \frac{|\sigma + \mathbb{L}W|^2}{\varphi^{\kappa+3}} \\ \mathrm{div}\,\mathbb{L}W = \frac{n-1}{n}\varphi^{\kappa+2}d\tau, \end{cases}$$

where all operators (and the scalar curvature) are defined with respect to the metric g.

- The first equation is called the **Lichnerowicz equation**,
- The second equation is the **vector equation**.

Outline

Some known results

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\ \varphi = -\frac{n-1}{n}\tau^2\varphi^{\kappa+1} + \frac{|\sigma + \mathbb{L}W|^2}{\varphi^{\kappa+3}} \\ \mathrm{div}\,\mathbb{L}W = \frac{n-1}{n}\varphi^{\kappa+2}d\tau, \end{cases}$$

1. **The CMC case :** In this case $d\tau = 0$ so the vector equation implies $W \equiv 0$ and one is left to solving the Lichnerowicz equation. The existence and uniqueness of a solution for the Lichnerowicz equation was settled by J. Isenberg (1995) :

   Except for specific cases, there exists a unique solution to the Lichnerowicz equation.

Some known results

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\ \varphi = -\frac{n-1}{n}\tau^2\varphi^{\kappa+1} + \frac{|\sigma + \mathbb{L}W|^2}{\varphi^{\kappa+3}} \\ \operatorname{div} \mathbb{L}W = \frac{n-1}{n}\varphi^{\kappa+2} d\tau, \end{cases}$$

1. **The CMC case :** In this case $d\tau = 0$ so the vector equation implies $W \equiv 0$ and one is left to solving the Lichnerowicz equation. The existence and uniqueness of a solution for the Lichnerowicz equation was settled by J. Isenberg (1995) :

   Except for specific cases, there exists a unique solution to the Lichnerowicz equation.

2. **The near CMC case :** it corresponds to the case where $\|d\tau\|$ is small compared to other quantities defined on M, it is usually addressed by perturbative arguments.

Some known results

1. **The CMC case :** In this case $d\tau = 0$ so the vector equation implies $W \equiv 0$ and one is left to solving the Lichnerowicz equation. The existence and uniqueness of a solution for the Lichnerowicz equation was settled by J. Isenberg (1995) :

   Except for specific cases, there exists a unique solution to the Lichnerowicz equation.

2. **The near CMC case :** it corresponds to the case where $\|d\tau\|$ is small compared to other quantities defined on M, it is usually addressed by perturbative arguments.

3. In 2008, M. Holst, G. Nagy and G. Tsogtgerel found a way to solve the conformal constraint equations on (M, g) such that $\mathcal{Y}(M, g) > 0$, τ arbitrary and σ small but non-zero.

Some known results

1. **The CMC case :** In this case $d\tau = 0$ so the vector equation implies $W \equiv 0$ and one is left to solving the Lichnerowicz equation. The existence and uniqueness of a solution for the Lichnerowicz equation was settled by J. Isenberg (1995) :

   Except for specific cases, there exists a unique solution to the Lichnerowicz equation.

2. **The near CMC case :** it corresponds to the case where $\|d\tau\|$ is small compared to other quantities defined on M, it is usually addressed by perturbative arguments.
3. In 2008, M. Holst, G. Nagy and G. Tsogtgerel found a way to solve the conformal constraint equations on (M, g) such that $\mathcal{Y}(M, g) > 0$, τ arbitrary and σ small but non-zero.
4. In 2010, M. Dahl, E. Humbert and R. G. discovered a new method to solve the conformal constraint equations called the **limit equation method**.

The limit equation method

Theorem (Dahl–G.–Humbert, G.–Sakovich)

If (M, g) satisfies $\mathrm{Ric} \leq -(n-1)g$, then, assuming further that $\tau > 0$ satisfies

$$\left\| \frac{d\tau}{\tau} \right\|_{L^\infty} < \sqrt{n},$$

the conformal constraint equations admit at least one solution (φ, W).

Outline

Conformal covariance

In the decomposition of $(\widehat{g}, \widehat{K})$ into $(g, \tau, \sigma, \varphi, W)$, we had to choose the metric g in the conformal class of $\widehat{g}$.

Question

How does the choice of g affects that of τ, σ, φ and W ?

Given two metrics $g, \widetilde{g} \in [\widehat{g}]$:

$$\widehat{g} = \varphi^{\kappa} g = \widetilde{\varphi}^{\kappa} \widetilde{g},$$

we set $\psi := \frac{\varphi}{\widetilde{\varphi}}$ so that $\widetilde{g} = \psi^{\kappa} g$. From

$$\widehat{K} = \frac{\tau}{n} \widehat{g} + \varphi^{-2}(\sigma + \mathbb{L}W) = \frac{\widetilde{\tau}}{n} \widehat{g} + \widetilde{\varphi}^{-2}(\widetilde{\sigma} + \mathbb{L}_{\widetilde{g}} \widetilde{W}),$$

we get $\tau = \widetilde{\tau}$ and

$$\psi^2 \widetilde{\sigma} - \sigma = \mathbb{L}W - \psi^{2+\kappa} \mathbb{L}\widetilde{W}.$$

Conformal covariance

$$\psi^2\widetilde{\sigma} - \sigma = \mathbb{L}W - \psi^{2+\kappa}\mathbb{L}\widetilde{W}.$$

- In the CMC case (constant τ), we have $W \equiv \widetilde{W} \equiv 0$ so

$$\widetilde{\sigma} = \psi^{-2}\sigma,$$

  Conformal covariance holds : (g, τ, σ) and $(\psi^\kappa g, \tau, \psi^{-2}\sigma)$ lead to the same solution(s).
- In the case $d\tau \not\equiv 0$, we do not expect $\psi^{2+\kappa}\mathbb{L}\widetilde{W}$ to be in the image of $\mathbb{L}$. If this were true for any ψ and $\widetilde{W}$, this would contradict the following proposition :

Proposition

Any $T \in \Gamma(\mathring{\mathrm{Sym}}_2)$ can be written as a finite sum $T = \sum_i f_i\, \mathbb{L}X_i$.

Conformal covariance

What we have shown is that York's decomposition is not conformally covariant !

But...

- There might be some black magic inside the conformal constraint equations that restores conformal covariance.
- The actual relation between σ and $\widetilde{\sigma}$ might not be the one we are expecting in the non-CMC case : $\widetilde{\sigma} \neq \psi^{-2}\,\sigma$.

Question

What would be a clear counterexample to show that the conformal method is not conformally covariant ?

Outline

Why should we care ?

- The conformal method has been highly successful in constructing CMC hypersurfaces and non CMC existence results are known.
- It is, amongst all known methods, by far the simplest one. No other method has produced such large class of solutions.
- If one insists on having conformal covariance, the **conformal thin sandwich method** is an extension of the conformal method that keeps track of the conformal changes.

Why should we care ?

- The conformal method has been highly successful in constructing CMC hypersurfaces and non CMC existence results are known.
- It is, amongst all known methods, by far the simplest one. No other method has produced such large class of solutions.
- If one insists on having conformal covariance, the **conformal thin sandwich method** is an extension of the conformal method that keeps track of the conformal changes.

Question

How can we tell whether two sets of seed data (g_1, τ_1, σ_1) and (g_2, τ_2, σ_2) lead to the same (set of) initial data ?

York's method B

York's splitting is not conformally covariant. But, we have decided to do the splitting w.r.t. the metric g, what if we do it w.r.t. the metric $\widehat{g}$:

$$\begin{aligned}\widehat{K} &= \frac{\tau}{n}\widehat{g} + \mathring{K}\\ &= \frac{\tau}{n}\widehat{g} + \widehat{\sigma} + \mathbb{L}_{\widehat{g}} W\\ &= \frac{\tau}{n}\widehat{g} + \varphi^{-2}\left(\sigma + \varphi^{\kappa+2}\mathbb{L}W\right)\end{aligned}$$

To be compared with $\widehat{K} = \frac{\tau}{n}\widehat{g} + \varphi^{-2}(\sigma + \mathbb{L}W)$ for method A (the conformal method). It leads to the following new system :

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\ \varphi = -\frac{n-1}{n}\tau^2\varphi^{\kappa+1} + \frac{|\sigma + \varphi^{\kappa+2}\mathbb{L}W|^2}{\varphi^{\kappa+3}} \\ \mathrm{div}(\varphi^{\kappa+2}\mathbb{L}W) = \frac{n-1}{n}\varphi^{\kappa+2}d\tau. \end{cases}$$

York's method B

York's method B is conformally covariant :

Proposition

(φ, W) is a solution to the previous system for $(g, \tau, \sigma) \Leftrightarrow (\widetilde{\varphi}, \widetilde{W})$ is a solution to the previous system for $(\widetilde{g}, \tau, \widetilde{\sigma})$ with

$$\widetilde{g} = \psi^{\kappa} g, \quad \widetilde{\sigma} = \psi^{-2}\sigma, \quad \widetilde{\varphi} = \psi^{-1}\varphi, \quad \widetilde{W} = W.$$

Observation

This new splitting gives rise to a projection map

$$\mathrm{proj}_B : (\widehat{g}, \widehat{K}) \to [g, \tau, \sigma].$$

Parameterizing the set of solutions to the constraint equations amount to understanding how the fiber $\mathrm{proj}_B^{-1}([g, \tau, \sigma])$ evolves when changing the base point $[g, \tau, \sigma]$. Redundancy is then suppressed.

Outline

Back to our last question

Question

What would be a clear counterexample to show that the conformal method is not conformally covariant ?

Back to our last question

Question

What would be a clear counterexample to show that the conformal method is not conformally covariant ?

Answer

If we can find a situation where seed data (g, τ, σ) lead to (at least) two distinct solutions (φ_1, W_1) and (φ_2, W_2), we get two initial data $(\widehat{g}_1, \widehat{K}_1)$ and $(\widehat{g}_2, \widehat{K}_2)$ and see how they decompose for another seed metric $\widetilde{g} \in [g]$.

Back to our last question

Question

What would be a clear counterexample to show that the conformal method is not conformally covariant ?

Answer

If we can find a situation where seed data (g, τ, σ) lead to (at least) two distinct solutions (φ_1, W_1) and (φ_2, W_2), we get two initial data $(\widehat{g}_1, \widehat{K}_1)$ and $(\widehat{g}_2, \widehat{K}_2)$ and see how they decompose for another seed metric $\widetilde{g} \in [g]$.

Indeed, in this case, there cannot be any well defined equivalence relation $\sim$ on the set of seed data so that

$$\mathrm{proj}_{\mathrm{A}} : (\widehat{g}, \widehat{K}) \to [g, \tau, \sigma]$$

is well defined.

The HNT-M method

In 2008, M. Holst, G. Nagy and G. Tsogtgerel introduced a new method to solve the conformal constraint equations. Their result was extended to the vacuum case by D. Maxwell shortly after :

Theorem (D. Maxwell, Nguyen T.C., G.–Ngô Q. A. ...)

Assume that (M, g) has positive Yamabe invariant : $\mathcal{Y}(M, g) > 0$. Then for any given τ, if $\|\sigma\|$ is small enough but $\sigma \not\equiv 0$, there exists at least one solution to the conformal constraint equations.

The proof is based on Schauder's fixed point theorem.

The HNT-M method

In 2008, M. Holst, G. Nagy and G. Tsogtgerel introduced a new method to solve the conformal constraint equations. Their result was extended to the vacuum case by D. Maxwell shortly after :

Theorem (D. Maxwell, Nguyen T.C., G.–Ngô Q. A. ...)

Assume that (M, g) has positive Yamabe invariant : $\mathcal{Y}(M, g) > 0$. Then for any given τ, if $\|\sigma\|$ is small enough but $\sigma \not\equiv 0$, there exists at least one solution to the conformal constraint equations.

The proof is based on Schauder's fixed point theorem. Alas...

Theorem (G. 2024)

The solution (φ, W) provided by this method is unique under a volume constraint :

$$\int_M \varphi^N d\mu^g = \mathrm{Vol}(M, \widehat{g}) \leq V_{\max}$$

for some given $V_{\max} > 0$.

Multiple solutions at last...

Surprisingly, this method continues to hold for manifolds with vanishing Yamabe invariant :

Multiple solutions at last...

Surprisingly, this method continues to hold for manifolds with vanishing Yamabe invariant :

Theorem (G. 2018, 2025(?))

Assume that (M, g) has vanishing Yamabe invariant : $\mathcal{Y}(M, g) = 0$. Then for any given τ, if $\|\sigma\|$ is small enough but $\sigma \not\equiv 0$, there exist 0, 1 or 2 solutions to the conformal constraint equations with volume less than V_{max}.

Multiple solutions at last...

Theorem (G. 2018, 2025(?))

Assume that (M, g) has vanishing Yamabe invariant : $\mathcal{Y}(M, g) = 0$. Then for any given τ, if $\|\sigma\|$ is small enough but $\sigma \not\equiv 0$, there exist 0, 1 or 2 solutions to the conformal constraint equations with volume less than $V_{\max}$.

The number of such solutions is the number of positive roots x to the following second order equation :

$$0 = \left[-\frac{n-1}{n}\int_M \tau^2 \varphi_0^{\kappa+2} d\mu^g + \int_M \frac{|\mathbb{L} W_0|^2}{\varphi_0^{\kappa+2}} d\mu^g\right] x^2 + 2x \int_M \frac{\langle \sigma, \mathbb{L} W_0\rangle}{\varphi_0^{\kappa+2}} d\mu^g + \int_M \frac{|\sigma|^2}{\varphi_0^{\kappa+2}} d\mu^g$$

Multiple solutions at last...

Theorem (G. 2018, 2025(?))

Assume that (M, g) has vanishing Yamabe invariant : $\mathcal{Y}(M, g) = 0$. Then for any given τ, if $\|\sigma\|$ is small enough but $\sigma \not\equiv 0$, there exist 0, 1 or 2 solutions to the conformal constraint equations with volume less than $V_{\max}$.

The number of such solutions is the number of positive roots x to the following second order equation :

$$0 = \left[-\frac{n-1}{n}\int_M \tau^2 \varphi_0^{\kappa+2} d\mu^g + \int_M \frac{|\mathbb{L} W_0|^2}{\varphi_0^{\kappa+2}} d\mu^g\right] x^2 + 2x \int_M \frac{\langle \sigma, \mathbb{L} W_0\rangle}{\varphi_0^{\kappa+2}} d\mu^g + \int_M \frac{|\sigma|^2}{\varphi_0^{\kappa+2}} d\mu^g$$

where φ_0 is the (normalized) zeroth eigenfunction of the conformal Laplacian

$$-\frac{4(n-1)}{n-2}\Delta\varphi_0 + \mathrm{Scal}\ \varphi_0 = 0 \qquad \text{and} \qquad \operatorname{div} \mathbb{L} W_0 = \frac{n-1}{n}\varphi_0^{\kappa+2} d\tau.$$

Construction of a counterexample

Note that, if $\mathrm{Scal} \equiv 0$, φ_0 is a constant (say $\varphi_0 \equiv 1$). Hence our equation becomes

$$0 = \left[-\frac{n-1}{n}\int_M \tau^2 d\mu^g + \int_M |\mathbb{L}X_0|^2 d\mu^g\right] x^2 + 2x \underbrace{\int_M \langle \sigma, \mathbb{L}X_0 \rangle d\mu^g}_{=0} + \int_M |\sigma|^2 d\mu^g,$$

i.e. its roots are symmetric w.r.t. 0 : these metrics do not provide the counterexample we need.

Idea

Fix a nice scalar flat metric (M, g_0) and numerically search for (g, τ, σ) with $g \in [g_0]$ so that our equation has two solutions.

Construction of a counterexample

Flat tori are not suitable candidates for (M, g_0) because they admit conformal Killing vector fields. But these vector fields are parallel. Instead we take (M, g_0) a suitable quotient of a flat torus. In dimension 3 there is only one suitable (oriented) choice, the **Hantzche-Wendt manifold** HW which is a quotient of $\mathbb{T}^3$ by $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein group) : its holonomy group leaves no vector invariant.

Further,

The covering $\pi : \mathbb{T}^3 \to HW$ is Galois.

This means that π^* maps tensors on HW **isomorphically** to G-invariant tensors on $\mathbb{T}^3$. Hence, together with the fact that we can do Fourier analysis on $\mathbb{T}^3$, we have very explicit L^2-orthonormal bases of all geometric tensor bundles.

Experimental results

Fact 1

There exist choices (g, τ, σ) such that our second order equation has either 0, 1 or 2 solutions and the transformation $(g, \tau, \sigma) \to (\psi^\kappa g, \tau, \psi^{-2}\sigma)$ changes the number of solutions.

Experimental results

Fact 1

There exist choices (g, τ, σ) such that our second order equation has either 0, 1 or 2 solutions and the transformation $(g, \tau, \sigma) \to (\psi^\kappa g, \tau, \psi^{-2}\sigma)$ changes the number of solutions.

So the natural conformal transformation is not a good equivalence relation on the seed data.

Fact 2

For seed data (g, τ, σ) for which there are two solutions, the corresponding two solutions $(\widehat{g}_1, \widehat{K}_1)$ and $(\widehat{g}_2, \widehat{K}_2)$ give different TT-tensors when decomposed with respect to another metric $\widetilde{g} \in [g]$:

$$(\widehat{g}_1, \widehat{K}_1) \text{ obtained from } (\widetilde{g}, \tau, \widetilde{\sigma}_1), \text{ but } (\widehat{g}_1, \widehat{K}_1) \text{ from } (\widetilde{g}, \tau, \widetilde{\sigma}_2),$$

with $\widetilde{\sigma}_1 \neq \widetilde{\sigma}_2$.

Fact 2

For seed data (g, τ, σ) for which there are two solutions, the corresponding two solutions $(\widehat{g}_1, \widehat{K}_1)$ and $(\widehat{g}_2, \widehat{K}_2)$ give different TT-tensors when decomposed with respect to another metric $\widetilde{g} \in [g]$:

$$(\widehat{g}_1, \widehat{K}_1) \text{ obtained from } (\widetilde{g}, \tau, \widetilde{\sigma}_1), \text{ but } (\widehat{g}_1, \widehat{K}_1) \text{ from } (\widetilde{g}, \tau, \widetilde{\sigma}_2),$$

with $\widetilde{\sigma}_1 \neq \widetilde{\sigma}_2$.

So there is no way to extend the conformal transformation $g \mapsto \psi^\kappa g$, $\tau \mapsto \tau$ to σ to obtain a well define quotient map $(\widehat{g}, \widehat{K}) \mapsto [g, \tau, \sigma]$.

Thank you for your attention !

Construction of a counterexample : algorithms

- We use the conformal thin sandwich method :

$$\begin{cases} -\frac{4(n-1)}{n-2}\Delta\varphi + \mathrm{Scal}\ \varphi = -\frac{n-1}{n}\tau^2\varphi^{\kappa+1} + \frac{\left|\sigma + \frac{1}{2N}\mathbb{L}W\right|^2}{\varphi^{\kappa+3}} \\ \mathrm{div}\left(\frac{1}{2N}\mathbb{L}W\right) = \frac{n-1}{n}\varphi^{\kappa+2}d\tau, \end{cases}$$

  This allows us to work with g "the" flat metric on HW and emulate conformal transformations by changing the "lapse function" $\frac{1}{2N}$.

- We decompose the objects according to the bases we exhibited :

$$\frac{1}{2N}(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3_+} a_{\vec{k}} c_{\vec{k}}(\vec{x}) + b_{\vec{k}} s_{\vec{k}}(\vec{x}),$$

$$\sigma(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3_+} \sum_i a^\sigma_{\vec{k}} c_{\vec{k},i}(\vec{x}) + b^\sigma_{\vec{k}} s_{\vec{k},i}(\vec{x}) \ldots$$

Construction of a counterexample : algorithms

- As $\frac{1}{2N}$ is a positive function, the equation

$$\operatorname{div}\left(\frac{1}{2N}\mathbb{L}W_0\right) = \frac{n-1}{n} d\tau$$

can be solved using the Choleski decomposition (spectral methods lead to dense matrices).

- We want that the seed data we find $(\frac{1}{2N}, \tau, \sigma)$ is close to a real solution : analytic functions have exponentially fast decaying Fourier coefficients.
- Hence, we set up a minimisation under constraint problem.

Construction of a counterexample : algorithms

We minimize

$$L := \sum_{\vec{k}\in\mathbb{Z}^3_+} e^{2\lambda|\vec{k}|}(a_{\vec{k}}^2 + b_{\vec{k}}^2) + \cdots ,$$

where $\frac{1}{2N}(\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3_+} a_{\vec{k}} c_{\vec{k}}(\vec{x}) + b_{\vec{k}} s_{\vec{k}}(\vec{x}), \ldots$ under the constraints

- Hard constraint : $a_{\vec{0}} \geq \mu \left(\sum_{\vec{k}\neq\vec{0}} |a_{\vec{k}}| + |b_{\vec{k}}|\right)$, $\mu < 1$, (has to be satisfied at each step of the minimisation procedure),
- Soft constraint :
  1. Normalisation : $\int_{HW} |\sigma|^2 d\mu^g = \int_{HW} |\tau|^2 d\mu^g = 1, a_{\vec{0}} = 1.$
  2. The second order equation $ax^2 + bx + c = 0$ ($c > 0$) has two positive roots : $b \leq -2\sqrt{ac} - \varepsilon$

- Due to the constraint $\int_{HW} |\sigma|^2 d\mu^g = 1$, σ is not small. We replace it by $\alpha\sigma$, $\alpha << 1$.
- The uniqueness statement in $(G.2024)$ shows that the shooting method

$$
\begin{cases}
\varphi_0 = x^{-1/(\kappa+2)}, \\
\operatorname{div}\left(\frac{1}{2N}\mathbb{L}W_{k+1}\right) = \frac{n-1}{n}\varphi_k^{\kappa+2} d\tau \\
-\frac{4(n-1)}{n-2}\Delta\varphi_{k+1} + \mathrm{Scal}\ \varphi_{k+1} = -\frac{n-1}{n}\tau^2\varphi_{k+1}^{\kappa+1} + \frac{\left|\sigma + \frac{1}{2N}\mathbb{L}W\right|^2}{\varphi_{k+1}^{\kappa+3}}
\end{cases}
$$

  actually works !
- Non-linearities are handled using Fast Fourier Transform algorithms. The classical rectangle quadrature formula for 1-periodic functions

$$\int_0^1 f(t)dt = \frac{1}{N}\sum_{k=0}^{N-1} f\left(\frac{k}{N}\right),$$

  is actually the optimal one (akin to Gauss quadrature formula).
- A better lattice for quadrature formula on HW actually exists but it is

Thank you once again for your attention !