Initial value problem in modified gravity

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Plan of the talk

- I. FLRW models and ODEs
- II. Well-posedness of the initial value problem
- III. Characteristic initial value problem for $f(R)$ theory

I. FLRW models and ODEs

Consider a flat FLRW model with a fluid $p_{\rm pf} = (\gamma_{\rm pf} - 1)\rho_{\rm pf}$ and a scalar field

$$\Box\phi = -\frac{dV}{d\phi} + \Gamma(\phi)u^{\mu}\partial_{\mu}\phi,$$

Get non-linear ODE system for the unknowns $\{a, H, \phi, \rho_{\rm pf}\}$:

$$\begin{aligned} \dot{a} &= aH, \\ \dot{H} &= -\frac{1}{2}\gamma_{\rm pf}\rho_{\rm pf} - \frac{\dot{\phi}^2}{2}, \\ \ddot{\phi} &= -(3H + \Gamma(\phi))\dot{\phi} - \frac{dV}{d\phi}, \\ \dot{\rho}_{\rm pf} &= -3\gamma_{\rm pf}H\rho_{\rm pf} + \Gamma(\phi)\dot{\phi}^2, \end{aligned}$$

together with the constraint

$$H^2 = \frac{\rho_{\rm pf}}{3} + \frac{\dot{\phi}^2}{6} + \frac{V(\phi)}{3},$$

- Complete description of solution space requires a global state space analysis
- This motivates globally covering state space adapted variables.

Recall the constraint

$$H^2 = \frac{\rho_{\rm pf}}{3} + \frac{\dot{\phi}^2}{6} + \frac{V(\phi)}{3},$$

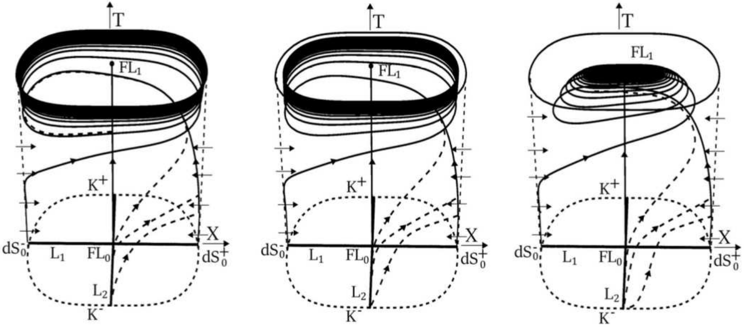
- Take $V(\phi) \sim \phi^2$ and introduce Hubble normalised variables:

$$\Omega_{\rm pf} = \frac{\rho_{\rm pf}}{3H^2}, \qquad \Sigma_\phi = \frac{\dot{\phi}}{\sqrt{6}H}, \qquad X = \frac{\lambda\phi}{\sqrt{6}H},$$

Then the constraint gives

$$\Omega_{\rm pf} = 1 - \Sigma_\phi^2 - X^2 \implies 0 < \Sigma_\phi^2 + X^2 < 1$$

By changing time to $0 < T < 1$, this gives a solid cylinder as phase-space.



(a) $\gamma_{\rm pf} - \langle\gamma_\phi\rangle > 0$. (b) $\gamma_{\rm pf} - \langle\gamma_\phi\rangle = 0$. (c) $\gamma_{\rm pf} - \langle\gamma_\phi\rangle < 0$.

An inflationary attractor solution exists and that it is associated with a 1-dim unstable center manifold of a de-Sitter fixed point, while the asymptotics are either governed by the massless scalar-field or can have a FLRW fixed point

Past results for scalar-tensor theories, torsion theories, $f(R)$ theories,....
(review in Bahamonde, Bohmer, Carloni, Copeland et al., Phys. Rep, 2018)

However Hubble normalised variables cannot cover the full phase-space.

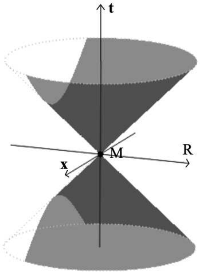
- Introduce new variables in the case $f(R) = R + \alpha R^2$

$$H = \sqrt{\frac{\alpha}{12}}(\mathrm{t} - \mathrm{x}), \quad \text{(2a)}$$

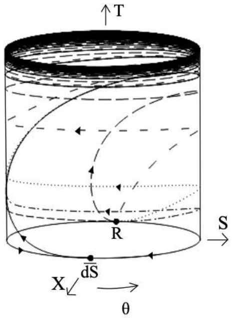
$$\dot{R} + RH + \frac{H}{2\alpha} = \frac{1}{\sqrt{12\alpha}}(\mathrm{t} + \mathrm{x}). \quad \text{(2b)}$$

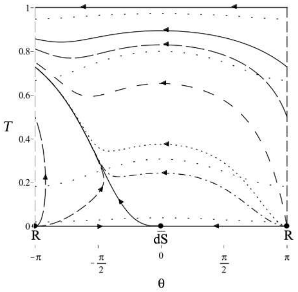
which brings the constraint equation to a quadratic canonical form:

$$\mathrm{t}^2 = \mathrm{x}^2 + R^2$$



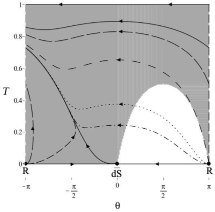
Results for $f(R) = \alpha R + \beta R^2$





Non-equivalence in the Jordan and Einstein frames $\tilde{g}_{ab} = f'(R) g_{ab}$ (due to positivity of conformal factor)

(Alho, Lima, FM, in preparation)
(Alho, Bessa, FM, J. Dyn. Diff. Eqs.,2025)



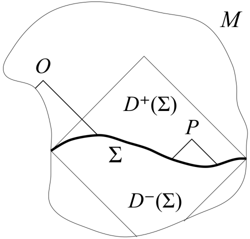
II. Well-posedness

Well-posedness of Cauchy problem

- Solutions exist and are unique?
- Continously depend on the initial data?

Initial data

- $(\Sigma, h_0, K_0, \text{fields})$
  k_0 is second fundamental form
- (Σ, h_0) Riemannian 3-manifold



Theorem (Choquet-Bruhat, 1952)

(Σ, h, K) initial data set satisfying the vacuum constraint equations. Then there exists a spacetime (M, g) locally in time satisfying the Einstein vacuum equations with Σ being a spacelike surface with induced metric h and second fundamental form K. The metric g depends continuously on the initial data. Its value at a point P depends only on the past of P. This local solution is unique.

$M = \Sigma \times \mathbb{R}$ such that the submanifolds $\Sigma_t = \Sigma \times \{t\}$ are spacelike.

$$g = -N(\theta^0)^2 + g_{ij}\theta^i\theta^j$$

where $\theta^0 = dt$ and $\theta^i = dx^i + \beta^i dt$ forms Cauchy adapted frame. Use $\mathcal{L}_0 = \partial_t - \mathcal{L}_\beta$ so

$$K_{ij} = \frac{1}{2}N^{-1}\mathcal{L}_0 g_{ij}$$

Use relation of Ricci tensor with second fundamental form to get 2nd order evolution equations for K_{ij} with principal part being the wave operator except for $\bar{\nabla}_i \partial K$. Other unkowns h and N appear at 2nd order except for $\mathcal{L}_0 \bar{\nabla}_j \partial_i N$.

To eliminate the bad terms use gauge choice:

$$\partial_0 N + N^2 K = 0$$

Get a quasi-linear equation

$$\Box_h K_{ij} = Q_{ij}(K, \partial K)$$

The local existence theorem of Leray for hyperbolic systems gives the local in time existence of solutions of this reduced system, in Sobolev spaces, with domain of dependence determined by the light cone. A solution of the reduced system is a solution of the full Einstein equations if the initial data is satisfied.

Developments:

- Generalised source fields: Maxwell, Yang-Mills, scalar fields, fluids,...
- Leray-Ohya non-strict hyperbolicity: fluids with conductivity (dissipation)

Question:

- Generalise to modified gravity?

Has torsion tensor S and metric compatible affine connection $C^{\gamma}_{\alpha\beta}$ such that

$$S_{\alpha\beta}{}^{\gamma} := C^{\gamma}_{[\alpha\beta]} := \frac{1}{2}\left(C^{\gamma}_{\alpha\beta} - C^{\gamma}_{\beta\alpha}\right) .$$

and can show

$$C^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + S_{\alpha\beta}{}^{\gamma} + S^{\gamma}{}_{\alpha\beta} - S_{\beta}{}^{\gamma}{}_{\alpha} ,$$

Why is torsion interesting?

- Accounts for quantum corrections within a geometric theory of gravity by relating the connection with matter intrinsic spin (Kibble, Sciama, 1962)
- Modifies the Buchdahl limit for the maximum compactness of static compact objects (Luz, Carloni, PRD, 2019)
- May prevent the formation of singularities (Luz, FM, J. Math. Phys. 2020)
- Even at very low densities, modifies cosmological dynamics, leading to characteristic gravitational waves or contributing to dark energy

Field equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \mathcal{T}_{\alpha\beta}$$
$$S_{\alpha\beta\gamma} + 2g_{\gamma[\alpha}S_{\beta]\mu}{}^{\mu} = -\Delta_{\alpha\beta\gamma}$$

Relation between hypermomentum and Lagrangian

$$\Delta_{\alpha\beta\gamma} = \frac{1}{\sqrt{-g}}\frac{\delta(\mathcal{L}_{\mathrm{matter}})}{\delta(C^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\alpha\beta})}$$

Example: The Weyssenhoff fluid

$$\Delta^{\alpha\beta\gamma} = \tau^{\alpha\beta}u^{\gamma} \quad \text{and} \quad S^{\alpha\beta\gamma} = -\frac{1}{2}\tau^{\alpha\beta}u^{\gamma}$$

where u^{α} is the fluid's 4-velocity and $\tau^{\alpha\beta}$ represents the spin

Applications reviewed by Hehl, arxiv, 2013

Generalises Choquet-Bruhat and York (C.R. Acad. Sci. Paris, 1995)

- We derive the Gauss-Codazzi equations including torsion
- We prove that the constraint equations are preserved during evolution
- Evolution equations written as a quasilinear system. Principal operator as a triangular matrix with characteristic polynomials with non-trivial multiplicity

Evolution equations (Luz & FM, arXiv, 2025)

$$
\begin{aligned}
-\mathrm{N}\Box_h K_{(ab)} &= Q_1(\nabla\nabla S, \nabla S, \nabla K) \\
-\frac{1}{\mathrm{N}}\mathcal{L}_0^2 S_{ab0} &= Q_2(\nabla S, S, K) \\
\frac{1}{\mathrm{N}}\mathcal{L}_0 S_{bi}{}^{i} &= Q_3(\nabla S, S, K) \\
\mathcal{L}_0 h_{ab} &= 2\mathrm{N}K_{(ab)} + 4\mathrm{N}S_{0(ab)}\ ,
\end{aligned}
$$

The system is non-strictly hyperbolic for the variables h_{ab}, $K_{(ab)}$, S_{ab0} and $S_{ai}{}^{i}$ in the sense of Leray-Ohya. If the Cauchy data in the Gevrey class of index 2, the Cauchy problem has a unique local solution.

- For vanishing torsion we recover the classical results for the Einstein equations

Physical motivation: Recall also talks of Laura Bernand and David Langlois

Scalar-tensor theories

$$S = \int \Big(F(\phi) R - H(\phi)(\nabla\phi)^2 - V(\phi) + L_{\text{matter}} \Big) dV$$

Brans-Dicke theory: Has $F(\phi) = \phi$ and $H(\phi) = \omega/\phi$ and $V(\phi) = 0$.

Horndeski theory: Lagrangian constructed from metric tensor and scalar field thats leads to 2nd order equations

$f(R)$ theories

$$S = \int \Big(f(R) + L_{\text{matter}} \Big) \, dV,$$

Gives 4th order differential equations:

$$f'(R)\, G_{ab} - \frac{1}{2}\Big(f(R) - R f'(R) \Big) g_{ab} + \left(g_{ab} \Box_g - \nabla_a \nabla_b \right) \left(f'(R) \right) = T_{ab}$$

Hyperbolicity in spherical gravitational collapse in a Horndeski theory

Justin L. Ripley and Frans Pretorius
Phys. Rev. D **99**, 084014 – Published 10 April 2019

Challenges to global solutions in Horndeski's theory

Laura Bernard, Luis Lehner, and Raimon Luna
Phys. Rev. D **100**, 024011 – Published 9 July 2019

Spherical collapse in scalar-Gauss-Bonnet gravity: Taming ill-posedness with a Ricci coupling

Farid Thaalba, Miguel Bezares, Nicola Franchini, and Thomas P. Sotiriou
Phys. Rev. D **109**, L041503 – Published 20 February 2024

Kinetic screening in nonlinear stellar oscillations and gravitational collapse

Miguel Bezares, Lotte ter Haar, Marco Crisostomi, Enrico Barausse, and Carlos Palenzuela
Phys. Rev. D **104**, 044022 – Published 9 August 2021

What can go wrong in generalised theories?

Observed that:

- Character of the system of equations changes from hyperbolic to elliptic (For open sets of initial data)
- Shocks and non-uniqueness
- Ostrogradski instabilities
- Characteristic speeds diverge
- Curvature and/or scalar field derivatives become large

Cauchy problem in generalised theories

- How to formulate a well-posed IVP? What is the initial data?
- If model starts "close" to GR, under what conditions remains "close" to GR?

Local well-posedness for scalar-tensor theories

(Cocke and Cohen, JMP 1968; Salgado, CQG, 2016; Avalos et al., JMP, 2018)

- Based on Choquet-Bruhat's approach

Local well-posedness for Horndeski theories

(Kovacs and Reall, PRL, 2020):

- Use generalised harmonic gauge $\Box x^\alpha = H^\alpha(x^i)$ and effective theory
- Higher derivative terms must be small compared to the 2-derivative terms (close to GR)
- Proved system is strongly hyperbolic and it follows in the L^2 norm

$$\| u(t,\cdot) \| \leq K e^{\alpha t} \| u_0(\cdot) \|$$

- Not true in the harmonic gauge

- Results also in Lovelock theories and scalar-Gauss-Bonnet (Bezares, Sotiriou at. al. PRD, 2025)

Local well-posedness for $f(R) = R + \kappa R^2$ with a massless scalar field

(LeFloch and Ma, Mém. Soc. Math. Fr., 2017):

- Use curvature as variable and small data close to GR.
- Follow Choquet-Bruhat and use energy estimates

Worked with Einstein metric $g = e^{2\rho}\tilde{g}$, used variable

$$\rho = \frac{1}{2}\ln f'(R)$$

and initial data $(g, \partial g, \rho, \partial\rho, \phi, \partial\phi)$ which is small in L^2 and asymptotically flat

$$\Box g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g, \rho, \partial\rho, \partial\phi)$$
$$\Box\phi = F(g, \partial\phi, \partial\rho)$$
$$\Box\rho = F(g, \rho, \partial\phi)$$

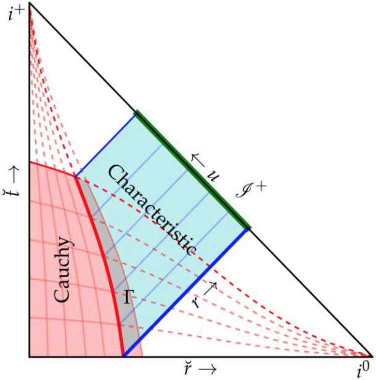
Global stability of Minkowski

(LeFloch and Ma, arXiv, 2024)

- Use $f(R) = R + \kappa R^2$ perturbations and Euclidean-Hyperboloidal foliations
- Exists globally hyperbolic Cauchy development approaching Minkowski when ID is sufficiently close to Minkowski

III. Characteristic initial value problem in $f(R)$

Characteristic evolution of gravitational waves in numerical relativity
Cauchy-characteristic extraction of waves (Barkett et al., PRD, 2020)



Caution: Some axi-symmetric systems are weakly hyperbolic and IVP is ill-posed in the L^2 norm (Hilditch et al., PRD, 2023)

Spherical symmetry with a massless scalar field and $\Lambda > 0$

(Costa, Duarte, FM, J. Hyp. Diff. Eqs, 2023)
Based on Christodoulou, CMP, 1986 for $\Lambda = 0$

Bondi metric

$$g = -f(u,r)\tilde{f}(u,r)du^2 - 2f(u,r)dudr + r^2\sigma_{\mathbb{S}^2}$$

Example: De-Sitter metric

$$g_{\mathrm{dS}} = -\left(1 - \frac{\Lambda}{3}r^2\right)du^2 - 2dudr + r^2\sigma_{\mathbb{S}^2}\ .$$

Einstein equations lead to a wave equation for the scalar field

$$\frac{1}{r}\left(\partial_u - \frac{\tilde{f}}{2}\partial_r\right)\partial_r(r\phi) = \frac{1}{2}(\partial_r\tilde{f})(\partial_r\phi)$$

and constraints

$$\frac{2}{r}\frac{1}{f}(\partial_r f) = (\partial_r\phi)^2$$

$$\partial_r(r\tilde{f}) = f(1 - \Lambda r^2)$$

So the problem is non-linear.

Penrose diagram for de Sitter spacetime

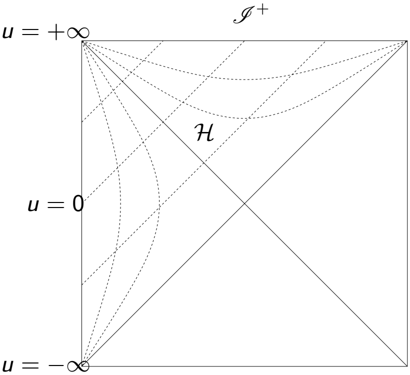


Figure: The dashed lines $u =$ constant are the future null cones of points at $r = 0$. The cosmological horizon $\mathcal{H}$ corresponds to $r_c = \sqrt{\frac{3}{\Lambda}}$ and future infinity $\mathscr{I}^+$ to $r = +\infty$.

Given the wave equation

$$\underbrace{\frac{1}{r}\left(\partial_u - \frac{\tilde{f}}{2}\partial_r\right)}_{D}\underbrace{\partial_r(r\phi)}_{h} = \underbrace{\frac{1}{2}(\partial_r\tilde{f})}_{G}(\partial_r\phi)$$

Content of Einstein's equations encoded in the integro-differential equation

$$\boxed{Dh = G\left(h - \bar{h}\right)}$$

IVP with data on a future light cone $u = 0$ with vertex at the center

$$\phi = \bar{h} := \frac{1}{r}\int_0^r h(u,s)ds$$

and, setting $f(u, r = 0) = 1$, the metric coefficients are given by

$$f(u,r) = \exp\left(\frac{1}{2}\int_0^r \frac{\left(h - \bar{h}\right)^2}{s}ds\right) \quad , \quad \tilde{f}(u,r) = \frac{1}{r}\int_0^r (1 - \Lambda s^2) f ds$$

Local existence: uses an iteration scheme

$$\begin{cases} D_n h_{n+1} - G_n h_{n+1} = -G_n \bar{h}_n \\ h_{n+1}(0, r) = h_0(r) \end{cases}$$

Integrate along the characteristics to construct a contracting sequence

$$h_{n+1}(u_1, r_1) = h_0(r_n(0)) e^{\int_0^{u_1} G_n dv} - \int_0^{u_1} \left(G_n \bar{h}_n \right) e^{\int_u^{u_1} G_n dv} du \, .$$

and show that there is δ and ε such that

$$\|h_{n+1} - h_n\| < \delta \quad \text{and} \quad \|h_n\| + \|\partial_r h_n\| < \varepsilon$$

Global existence: uses energy estimates

$$\mathcal{E}(u) = \sup \left|(1 + r)^{2-\delta} \partial_r h(u, r)\right|,$$

- Also got existence results for massive scalar fields with potential $V(\phi) \sim \phi^2$

(LeFloch, FM, Nguyen, J. Diff. Eqs., 2024)

Global existence and uniqueness

Let $\phi_0(r) \in C^{k+1}$ and

$$\sup_{r\geq 0}\left(|\partial_r(r\phi_0)| + |(1+r)^2\partial_r^2(r\phi_0)| + |(1+r)^3\partial_r^3(r\phi_0)|\right) < \infty$$

There is $\epsilon_0 > 0$ such that when

$$\sup_{r\geq 0}|(1+r)^2\partial_r^2(r\phi_0(r))| < \epsilon_0$$

then there exists a unique global C^k solution satisfying the initial data

Frame choice and metric convergence

Fix Bondi time at infinity $d\hat{u} = f(u, r=\infty)du$. Then, there exists orthonormal frame $(e_I)_{I=0,1,2,3}$ such that, by writing $g_{IJ}(\hat{u}, r) = g(e_I, e_J)$ we have

$$|g_{IJ}(\hat{u}, r) - g_{IJ}^{dS}| \lesssim \frac{1}{(1+r)^2}e^{-2(1-\varepsilon)H_\delta \hat{u}}$$

and the spacetime is geodesically complete towards the future.

(LeFloch & FM, to appear soon)

Consider

$$S[\phi, g] =: \int_M \Big(f(R) + L[\phi, g]\Big)\, dV,$$

and

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \Big(\frac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi)\Big)g_{ab},$$

Consider $\tilde{g} = e^{\kappa\rho}g$ and use variables

$$\rho = \frac{1}{\kappa}\ln f'(R), \qquad l = \partial_r(r\rho), \qquad h = \partial_r(r\phi), \qquad \bar{h} = \phi, \qquad \bar{l} = \rho$$

For $\tilde{g} = -e^{2\nu}du^2 - 2e^{\nu+\lambda}dudr + r^2 g_{S^2}$, get system

$$Dh = \frac{1}{2r}G(h - \bar{h}) - \frac{r}{2}e^{\nu+\lambda-\kappa\bar{l}}V'(\bar{h}) + (\dots)$$
$$Dl = \frac{1}{2r}G(l - \bar{l}) - \frac{r}{\kappa}e^{\nu+\lambda}\Big(W_\star(\bar{l}) - \frac{16\pi}{3}e^{-2\kappa\bar{l}}V(\bar{h})\Big) + (\dots),$$

- Need extra regularity condition at the origin
- Obtain local well-posedness for asymptotically flat C^1 initial data

Hawking mass function is

$$m(u,r) = \frac{r}{2}\left(1 - e^{-2\lambda}\right).$$

For $Rf'(R) > f(R)$ and $V(\phi) \geq 0$ we can show

$$1 - \frac{2m}{r} = \frac{1}{re^{\nu+\lambda}} \int_0^r \left(1 - s^2 e^{-2\kappa\bar{l}}\left(V_\star(\rho) + 8\pi V(\phi)\right)\right) e^{\nu+\lambda} ds \in (0,1).$$

The mass is a non-decreasing function in radial directions and non-increasing function along the incoming light rays towards the future:

$$\partial_r m \geq 0 \quad \text{and} \quad Dm \leq 0.$$

These properties played important role in the global existence of solutions of the Einstein-massive scalar field system

Assume the existence of the limit $M_0 := \lim_{r \to +\infty} m(0, r)$, that is, the initial total mass is finite. Denote the *Bondi mass* by

$$M(u) := \lim_{r \to +\infty} m(u, r)$$

Then we have the following result parallel to the classical Bondi theorem:
The Bondi mass M is a monotonically non-increasing function of the variable u:

$$\frac{dM}{du} \leq 0.$$

Consider a sufficiently regular and globally defined solution to the system.
From previous properties:

$$M_f = \lim_{u \to +\infty} M(u)$$

exists, which is referred to as the *final Bondi mass*.

- For $r_0 > 2M_f$, the timelike lines $r = r_0$ are complete towards the future.

- Modified gravity brings new mathematical challenges
- There are few rigorous maths results about well-posedness of IVP

- Constructed a global and regular dynamical system for some $f(R)$ theories

- Obtained local well-posedness result for Einstein-Cartan theory
- Built first-order characteristic IVP for $f(R)$-scalar field spherical systems
- Obtained monotonicity properties for the mass

Future work

Global solutions, large data, gravitational collapse, trapped surfaces,....